Lecture 12

Forced Oscillations

12.1 Forcing equations

Because of damping, no oscillation would continue indefinitely in the absence of some sort of outside force. In general, we've considered outside forces only in terms of initial conditions that the equations must satisfy, but we can also put external forces directly into the equations. We will consider only a general force, that varies as $F_0 \cos \omega t$, where now ω and δ are arbitrary values that are unrelated to the natural oscillation frequency. Once we have solved the problem for a cosine force, we have, in effect, solved it for any force, as any arbitrary time-varying force can be decomposed into a Fourier series of cosines and sines.

We will again write everything in terms of a force balance equation, as dealing with potentials in a system with constant energy input is too horrible to even consider. The general equation should look some like mass $\times$acceleration equals the spring force minus the damping force plus the driving force, or

$$m\frac{dx^2}{dt^2} = -kx - \gamma\frac{dx}{dt} + F_0 \cos \omega t,$$

or, rewritten the way we like it best,

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t$$

where we have used the definition that $\omega = \sqrt{k/m}$.

This equation is a second order, linear, but *non*-homogeneous differential equation. A homogeneous equation is one where there are no constant terms, like the damped oscillator equation that we solved previously. In a non-homogenous equation, the constant term prevents us from just solving the characteristic equation to find the solution to the equation.

Once we solve the equation, however, it is obvious that the solution to the *homogenous* equation can be added to the solution in arbitrary amounts. That is, there will be some solution that we get to the forcing equation, but there will also be a component that looks just like the non-forced equation. The solution to the non-forced equation,

$$x_h(t) = A\exp(\frac{-\gamma}{2m}t)\cos[t\sqrt{\omega^2 - \gamma^2/4m^2} + \phi],$$

dies exponentially, and will eventually be gone completely, leaving only the forced solution left. We will call the non-forced solution the *transient* part of the solution and the forced solution the *steady-state* part of the solution. The two parts of the solution are completely independent and non-interacting.

How do we solve the non-homogeneous part of the equation? There is no general method to solving such a thing except by making the assertition or observation or guess that the solution oscillates with the same frequency as the forced solution, but perhaps with a phase shift. This guess would imply a solution of the form

$$x_n(t) = B\exp[i(\omega t - \delta)]$$

where, again, ω is thea frequency of the forcing, rather than ω_0, the natural frequency that the spring would like to oscillate instead.

We could do this whole equation out with imaginary numbers and then take the real part at the end, but, in this case, it actually turns out to be easier to do things explicitly in the real domain, so we have instead that

$$x_n(t) = B\cos(\omega t - \delta)$$

which we can substitute in to find

$$-\omega^2 B\cos(\omega t - \delta) - \omega\frac{\gamma}{m}B\sin(\omega t - \delta) + \omega_0^2 B\cos(\omega t - \delta) = F\cos(\omega t)$$

We can expand out the sums to get

$$-\omega^2 B\cos\omega t\cos\delta - \omega^2 B\sin\omega t\sin\delta - \omega\frac{\gamma}{m}B\sin\omega t\cos\delta + \omega\frac{\gamma}{m}B\cos\omega t\sin\delta + \omega_0^2 B\cos\omega t\cos\delta + \omega_0^2 B\sin\omega t\sin\delta - F\cos\omega t = 0$$

or

$$\begin{aligned}(-\omega^2 B\cos\delta + \omega\frac{\gamma}{m}B\sin\delta + \omega_0^2 B\cos\delta - F) \quad &\cos\omega t \\ +(-\omega^2 B\sin\delta - \frac{\gamma}{m}\omega B\cos\delta + \omega^2 B\sin\delta) \quad &\sin\omega t \quad = 0\end{aligned}$$

This expression must be zero for all times t, which is only possible if the terms in parenthesis are independently equal to zero. This gives us two independent equations that we can use to solve for B and δ, the two parameters for which we seek values.

From the $\sin \omega t$ term, we can solve for δ

$$(\omega_0^2 B - \omega^2 B) \sin \delta = \omega \frac{\gamma}{m} B \cos \delta$$

or

$$\tan \delta = \frac{\omega \gamma / m}{\omega_0^2 - \omega^2}$$

which is the phase delay associated with the forcing. For example, if $\omega > \omega_0$, δ is negative, if $\omega < \omega_0$, δ is positive, and if $\omega \sim \omega_0$, $\delta \sim 90$ degrees.

We will get B from the $\cos \omega t$ term:

$$B(-\omega^2 \cos \delta + \omega \frac{\gamma}{m} \sin \delta + \omega_0^2 \cos \delta) = F$$

which gives

$$B = \frac{F}{(\omega_0^2 - \omega^2) \cos \delta + \omega \frac{\gamma}{m} \sin \delta},$$

into which we can substitute the fact that

$$\sin \delta = \frac{\omega \gamma / m}{\sqrt{\omega^2 \gamma^2 / m^2 + (\omega_0^2 - \omega^2)^2}}$$

and

$$\cos \delta = \frac{\omega_0^2 - \omega^2}{\sqrt{\omega^2 \gamma^2 / m^2 + (\omega_0^2 - \omega^2)^2}}$$

which gives us finally

$$B = \frac{F}{\frac{(\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2)}{\sqrt{\omega^2 \gamma^2 / m^2 + (\omega_0^2 - \omega^2)^2}} + \frac{\omega^2 \gamma^2 / m^2}{\sqrt{\omega^2 \gamma^2 / m^2 + (\omega_0^2 - \omega^2)^2}}}$$

or

$$B = \frac{F}{\sqrt{\omega^2 \gamma^2 / m^2 + (\omega_0^2 - \omega^2)^2}}$$

Note now that B, which is the amplitude of oscillation, is *not* a constant of integration. It is a parameter solely determined by the details of the system. The initial conditions go only into the transient solution, not into the forced solution!

We now define a new parameter, which will turn out to be the inverse of the fraction of energy lost per oscillation, called the "quality factor" or just Q, where

$$Q = \frac{m \omega_0}{\gamma}$$

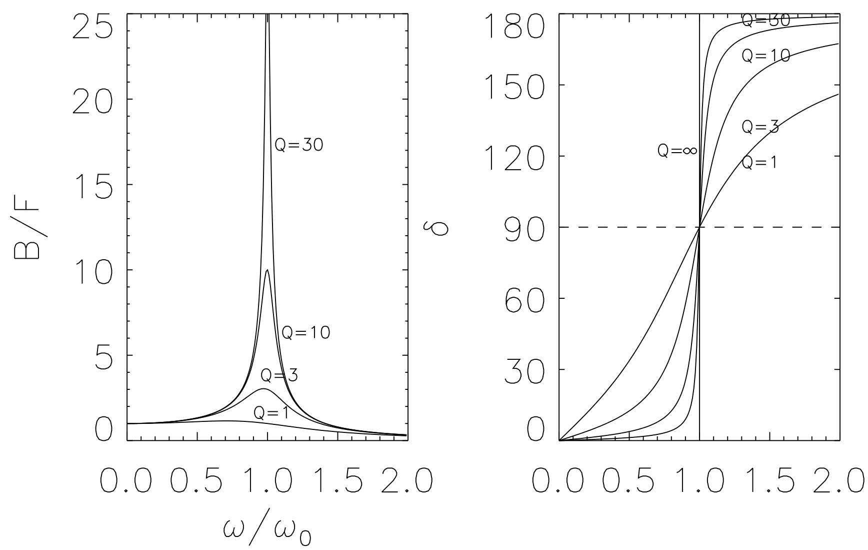

Figure 12.1: The response of an oscillator to a forced oscillation. The natural freqency of the oscillator is ω_0, while the forced oscillation occurs at frequency ω. The first plot shows the magnitude of the forced oscillation, compared to the magnitude of the forcing, while the second shows the phase difference between the forcing and the oscillation, for different values of the quality factor Q.

which, when substituted into the solution, gives us the full steady-state solution to the forced equation:

$$x_n(t) = \frac{F \cos\left[\omega t - \tan^{-1}\left(\frac{\omega\omega_0}{Q(\omega_0^2-\omega^2)}\right)\right]}{\sqrt{(\omega_0^2-\omega^2)^2 + \frac{\omega^2\omega_0^2}{Q^2}}}$$

The response of the forced oscillator is best illustrated graphically, as in Figure 12.1.

What is the meaning of this equation? As can be seen from the equation, "high quality" oscillators turn a small forcing into a huge amplitude in the vicinity of their natural frequency, while low quality oscillators never acheive high amplitude oscillations, no matter what the forcing frequency. What makes an oscillator high or low quality? From the definition of Q is can be seen that Q is inversely related to the damping, so that highly damped systems will be low quality, and never achieve high amplitude, while lightly damped systems will be high quality oscillators.

Imagagine now that we think of oscillations of not just springs and pendula, but also the vibrations of cars, bells, buildings, and anything else. A bell has a natural frequency just like a spring; it is the tone that is heard when the bell is struck. But the bell does not need to be forced at that particular frequency to hear the tone, it just needs to be hit with some random force. Why is this? Again, because of a Fourier series decomposition. And force hitting the bell can be decomposed to a series that includes a cosine term at the resonant frequency. If Q is high, the amplitudes of all of the other frequencies will be quite small compared to the amplitude of the resonance frequecy, and all you will hear is one tone. If Q is high, γ is low, so the damping is low, and the tone will continue to sound for a long time, implying a high quality oscillator.

When I drive my car over 80 miles per hour, something in the dashboard starts to rattle. WHen I slow down, it quits rattling, and when I speed up, it also quits rattling. What can we learn from these facts? First, there must be something about going 80 mph that causes forcing at a particular frequency, and whatever frequency this is, there is something in the dashboard for which this is the natural vibration frequency. We can also tell, since the vibration goes away pretty quickly when we're not at the right speed, that the damping must be pretty high and therefore the Q is low.

AS a kid, I tried to knock down the Gateway Arch in St. Louis by running back and forth inside of it as fast as I could. I figured that I could get it oscillating and then make it collapse. I'm not sure how I planned to escape the wreckage. But, luckily for me, I could never get it to move (there was an accelerometer or something at the top so you could actually see the movements). Why not? Well, for one, I was probably not at the resonant frequency, so nothing was every likely to happen. Also, importantly, the oscillations in the arch were highly damped, so even if I *had* found the right frequency, nothing much would have happened. If this were not the case, *something* would likely have accidentally found the resonant frequency – a blowing wind, the rumbles of the train tracks – and would have caused huge oscillations in the building. The most famous example of this resonance being found, is, of course, the Tacoma Narrows bridge, which fell to pieces spectacularly one day when a light gale force wind happened to chance upon the resonant frequency of the bridge.

What is the resonant frequency of the shocks in your car? You can find it by pushing down on the bumper and estimating the oscillation frequency that results. The result is probably something very much like 1 Hz or thereabouts. So if you find a bumpy road and drive at such a speed so that you are hitting bumps at about 1 per second, you can get your car to resonantly oscillate. Try this at home. But if you now drive faster, the amplitude of the car's vibrations decrease, and, as you can see from Figure 12.1, the faster you go (i.e. the higher the forcing frequency) the lower amplitude the vibrations are. So if driving on washboard, could you drive really really fast to keep from getting a

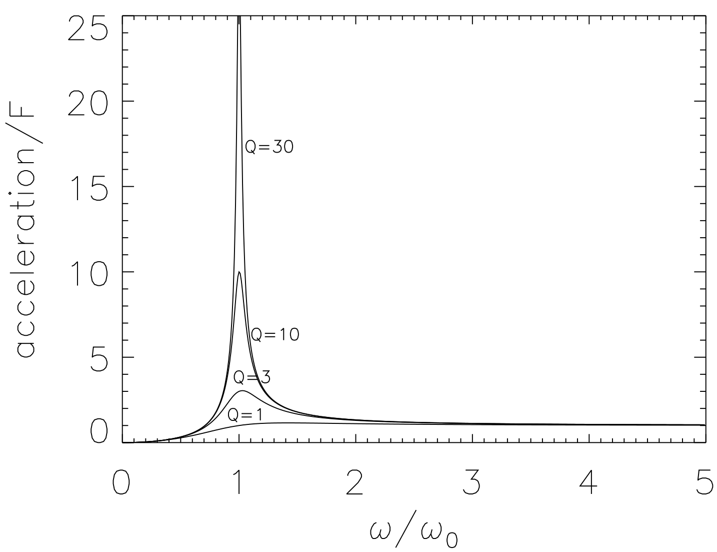

Figure 12.2: Acceleration experience by a forced oscillator.

bumpy ride? The answer is yes, but only within limits. Because the "bump" you feel from a ride is not the total magnitude of the oscillation, but rather the maximum acceleration experienced during the ride. For a motion going as $x = B\cos\omega t$, the maximum acceleration occurs at the inflection points and is $\frac{d^2x}{dt^2} = \omega^2 B$. Using our solution for B above and assuming that $\omega \gg \omega_0$, we find that $a = F$. That is, at high frequency the acceleration is just equal to the forced acceleration. If we now plot the acceleration as a function of ω/ω_0 we get the results in Figure 12.1, which show that although accelerations can be large around resonances, as the forcing goes to higher frequency the acceleration drops to an asymotopic value regardless of Q. So, indeed, as long as you drive a couple of times faster than the resonance frequency, your accelerations stay to a minimum, and though they won't get much lower by driving faster, they also won't get any higher by driving faster. Nonetheless, the lowest acclerations are still experienced by going much slower than the resonance instead.

12.2 Oscillation Lab

We are now going to the Tournament Park Oscillation Lab to do experiments in oscillation.