Lecture 9

Gravitation

9.1 Newton's law

There is no more common force in our experience than that of gravity. It determines the details of how an object falls to the ground as well as how a gigantic planet like Jupiter moves in orbit around the sun and even how our galaxy rotates around itself. The fundamental physics behind all of these motions remains Newton's empirically determined law of gravitation: two bodies of whatever masses attract each other along the line connecting them with a force proportional to the product oftheir mass and inversely proportional to the square of the separation distance. If m_1 and m_2 are the masses of the objects,

$$F = -G\frac{m_1 m_2}{r^2},$$

where G is Newton's gravitational constant, F is the attractive force, and r is the distance between the objects.

This simple law applies to every object in the universe: all objects attract all objects. These days, we know that General Relativity modifies gravitation somewhat, but, for almost all terrestrial and even solar system applications, Newton's law can be said to hold exactly.

If every object attracts every other object, then a distant star attracts us as does a nearby cup of coffee. Using Newton's laws, we can even determine the relative pull we feel from the two. The nearest star is Alpha Centuri, at 4.4 light-years, or 4.1×10^{16} m distance. It's mass is about $2 \times 10^{3}0$ kg. A cup of coffee (minus the cup), in contrast, weighs in at approximately a quarter of a kilogram. How distant does the cup of coffee have to be to attract as much as Alpha Centuri? The forces should be equal, so we have

$$G\frac{m_{\text{us}} m_{\text{coffee}}}{r^2_{\text{coffee}}} = G\frac{m_{\text{us}} m_{\text{star}}}{r^2_{\text{star}}}$$

or

$$r_{\text{coffee}} = r_{\text{star}}\sqrt{\frac{m_{\text{coffee}}}{m_{\text{star}}}}$$

which turns out to be about 15 meters.

Gravity (and indeed all of the fundamental forces of nature) is a *linear* force, meaning that the force due to a combination of objects is simply the sum of the individual forces (where we have to sum in a vector sense: the forces are all directional). Thus because we now know the magnitude of the force due to Alpha Centari and our cup of coffee, we could place the cup of coffee 15 m away from us on the side opposite of Alpha Centari and the two forces would add together and cancel each other out. In addition to α Cen, billions and billions of other stars and galaxies exist, each attracting us (and us attracting them) separately, and each of their contributions can be added up separately to determine their total pull (on average, of course, the universe is the same in all directions, so the forces from opposite directions cancel and we are left floating, at least on average).

Let's now look at the general three-dimensional case with an arbitraty coordinate system. Object 1 is at (vector) position $\mathbf{r}_1$, object 2 at $\mathbf{r}_2$, and the separation between the two is $\mathbf{r}_1 - \mathbf{r}_2 \equiv \mathbf{r}_{12}$. Newton's law then becomes

$$\mathbf{F} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12},$$

where now $|\mathbf{r}_{12}|$ is the magnitude of the vector $\mathbf{r}_{12}$, and $\hat{\mathbf{r}}_{12}$ is the direction of $\mathbf{r}_{12}$ (but with unit magnitude – it just means that the force is in the direction of the vector). You will sometimes see this written as

$$\mathbf{F} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12},$$

which is mathematically identical, but can lead to a bit of confusion because it now looks like the force is proportional to the cube of the distance. We'll use the first form exclusively.

For a collection of n objects, the force felt is the sum

$$\mathbf{F} = -G\sum_{1}^{n}\frac{m_0 m_n}{|\mathbf{r}_n - \mathbf{r}_0|^2}\hat{\mathbf{r}}_n$$

or, for a continuous density distribution, the force felt at position $\mathbf{r}_0$ is

$$\mathbf{F}(\mathbf{r_0}) = -G\int_{\text{V}}\frac{m\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|^2}(\widehat{\mathbf{r}-\mathbf{r_0}})dV$$

9.2 Force from the Earth

In most terrestrial applications, we won't use an arbitrary coordinate system, we'll use one centered on the earth, and one of the masses will be the earth itself, which we'll now denote M. If $\mathbf{r}$ is now the vector from the center of the earth, then

$$\mathbf{F} = -G\frac{Mm}{\mathbf{r}^2}\hat{\mathbf{r}},$$

and $\hat{\mathbf{r}}$, of course, points away from the center of the earth (which is why the force is negative, i.e. towards the earth.

The earth, however, is far from uniform. The density of the core is much higher than that of the mantle, and the crust is lightest of all. How does this affect what we feel at the surface? Because gravity is linear, the contributions from each part of the earth can be considered separately and then added together, using the integral form of the equation above.

First, for an ideal spherical earth, we should switch to spherical coordinate where $\mathbf{r}$is not given by (x, y, z), but rather by (r, θ, ϕ), where r is the distance from the center of the earth, θ is the projected latitude, and ϕ is the projected longitude. To good approximation, although the density of the earth varies greatly in r, it varies little in θ or ϕ, which is to say that the density is almost constant within any small shell of material at radius r. There are clearly deviations to this approximation, and these deviations give us some of our best information about the interior of the earth.

So we want to integrate the density throughout the earth to calculate the gravitational force at the surface. To do the integral in spherical coordinates, though, requires transforming the differentials dx, dy, and dz in the volume integral to radial coordinates with the use of the *Jacobian*, which you recall from calculus. Spherical coordinate systems are often useful, so it pays to remember the general form of the volume integral transformation:

$$\int dx \int dy \int dz = \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\phi \int r^2 dr.$$

Thus the force felt at the surface is just equal to

$$\mathbf{F}(\mathbf{r_0}) = -Gm \int_0^{\pi} \sin(\theta)\ d\theta \int_0^{2\pi} d\phi \int \frac{r^2\rho(r,\theta,\phi)}{|\mathbf{r}-\mathbf{r}_0|^2}(\widehat{\mathbf{r}-\mathbf{r_0}})dr$$

Now let's consider the contribution of the gravity from just one shell of thickness dr at radius r. We are assuming that there are no azimuthal or polar variations in the density, so $\rho(r, \theta, \phi) \equiv \rho(r)$. What is $|\mathbf{r} - \mathbf{r}_0|^2$? For simplicity, let's put outselves at the north pole, since all positions on the earth should be equivalent, then the distance should depend only on r and θ, as seen in Figure 9.2.

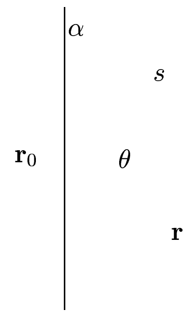

Figure 9.1: Geometry for the force from a thin shell. The angle θ is the polar angle, with $\theta = 0$ being the north pole, r is the radius of the thin shell, and r_0 is the distance from the observer to the center of the thin shell.

We can rewrite the integral for the force associated with a single thin shell as

$$\mathbf{F}_{\mathrm{shell}} = -Gm\rho(r)r^2 dr \int_0^{\pi} \sin(\theta)\ d\theta \int_0^{2\pi} d\phi \frac{1}{s^2} \widehat{(\mathbf{r_0} - \mathbf{r})}.$$

From the symmetry of the problem, we can see that as we integrate around ϕ all contributions to the vector other than in the $\hat{\mathbf{r}}$ direction will cancel, so that $\mathbf{F}$ will only be in the radial direction (which agrees with our experience that gravity is towards the center of the earth). The radial component of vector $\widehat{(\mathbf{r_0} - \mathbf{r})}$ has magnitude $\cos\alpha = (r_0 - r\cos\theta)/s$, so we can finally rewrite the integral as

$$\mathbf{F}_{\mathrm{shell}} = -2\pi Gm\rho(r)r^2 dr\ \hat{\mathbf{r_0}} \int_0^{\pi} sin\theta \frac{r_0 - r\cos\theta}{s^3} d\theta.$$

The distance can be determined simply from the law of cosines,

$$s^2 = r_0^2 + r^2 - 2r_0 r\cos\theta,$$

and now we make life easier by playing a few tricks. First, from the previous eqaution, we can substitute

$$r\cos\theta = \frac{r_0^2 + r^2 - s^2}{2r_0}$$

and we can also write the differential

$$2sds = 2r_0 r\sin\theta d\theta$$

Substituting both of these in, we now have:

$$\mathbf{F}_{\text{shell}} = -2\pi Gm\rho(r)r^2 dr\ \hat{\mathbf{r_0}} \int_{r_0-r}^{r_0+r} \frac{r_0^2 - r^2 + s^2}{2r_0^2 r s^2} ds$$

or

$$\mathbf{F}_{\text{shell}} = -\frac{\pi Gm\rho(r)r}{r_o^2} dr\ \hat{\mathbf{r_0}} \int_{r_0-r}^{r_0+r} \left(\frac{r_0^2 - r^2}{s^2} + 1 \right) ds,$$

which we can now integrate to be

$$\mathbf{F}_{\text{shell}} = -\frac{\pi Gm\rho(r)r}{r_o^2} dr\ \hat{\mathbf{r_0}} \left(-\frac{r_0^2 - r^2}{s} + s \right) \Bigg|_{r_0-r}^{r_0+r}. \tag{9.1}$$

After a little algebra, the integral turns out to reduce simply to $4r$. The total mass of the shell is $4\pi\rho(r)dr$ (surface area times thickness times density equals mass), which we will write as m_{shell}, so the total force from the shell is

$$\mathbf{F}_{\text{shell}} = -\frac{Gmm_{\text{shell}}}{r_0^2},$$

but this is exactly the same force that would be felt from a point mass of total mass m_{shell} situated at the center of the earth. Put another way, we have the vitally important result: *a uniformly dense spherical shell attracts an external mass as if all of its mass were concentrated at its center.* And since a sphere can be considered to be a collection of uniform shells, the sphere attracts as if all of its mass were located at the center of the sphere.

One interesting consequence of the fact is a question that the random person on the street invariably gets wrong: what would happen to the orbit of the moon if the earth were compressed until it turned into a black hole? The answer, of course, is that the moon only cares about the mass of the earth, not the radial distribution of that mass, so the moon would stay happily in its orbit.

What would happen if we were *inside* the shell? The limits in equation 9.1 would then be changed (why?) and we would have instead

$$\mathbf{F}_{\text{shell}} = -\frac{\pi Gm\rho(r)r}{r_o^2} dr\ \hat{\mathbf{r_0}} \left(-\frac{r_0^2 - r^2}{s} + s \right) \Bigg|_{r-r_0}^{r_0+r} = 0.$$

The force from each direction would cancel and no force would be felt! This result is obvious if we were at the center of the shell, but it holds no matter where in the shell we are.

One consequence of these two results is that the gravitational pull of the earth depends only on the mass within a sphere at the radius being considered. If we go halfway to the center of the earth we only feel the pull of the shells inside of us, and all of those above us give zero net pull. Assuming that the earth is constant density, we can calculate the gravitational pull from the mass,

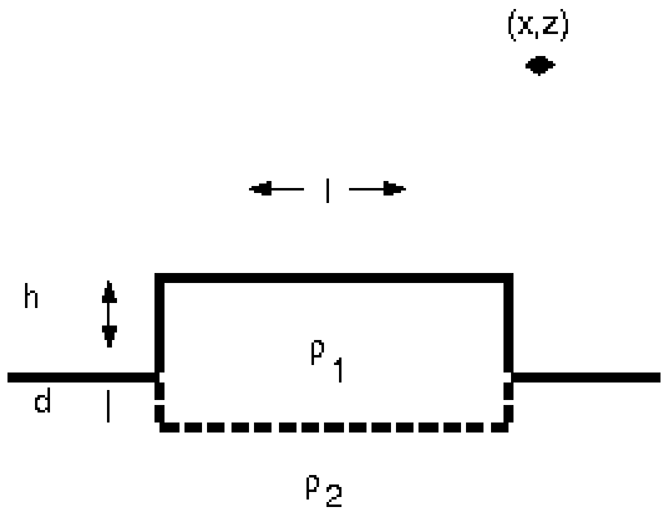

Figure 9.2: An isostatically compensated mountain range. The nonuniformity of the density field causes a gravitational anomaly.

which is $1/2^3 = 1/8$ of the total mass of the earth and the square of the distance from the center, which is $1/2^2 = 1/4$ of that at the surface of the earth, so the total force felt is $1/2$ of that felt at the surface. In real life, of course, the earth is not constant density. The crust is much lighter than the mantle which is lighter than the core, so a sphere with $1/2$ the radius of the earth contains much more than $1/8$ the mass of the earth, meaning that the true gravity would be greater. In fact, because the mantle is more dense than the crust, as we descend through the crust, the earth's gravity actually *increases* for a while before it then starts decreasing!

9.3 Gravitational anomalies

The earth is neither truly a sphere nor uniform but it is close enough to both that we can use the previous approximations for large scale information. At smaller scales, the deviations from uniformity contain interesting geological information. This information is extracted by a method called *gravity mapping*, where an instrument is carried around the surface of the earth (or in the air or in space, really), the gravity field is measured, and deviations from the field are mapped. Places where the gravity differs from what is expected for a uniform earth are called *gravitational anomalies*. These anomalies are used to tell us about structures beneath the surface that we couldn't otherewise see.

Consider a simplistic case, as illustrated in Figure 9.3, where we have a square mountain range of height h, length and width l, and, for simplicity, infinite length. The mountain range is made of crustal material of density ρ_1, and it sits on mantle material of densty ρ_2. We have been taught that such a mountain range is essentially floating on the denser mantle material and is *isostatically compensated* by a root of material that penetrates below the mantle, but how do we really know this is true? Let's now calculate the gravitational perturbation caused by this mountain range. First off, because gravity is linear, we can calculate the total gravitational force simly calculating the force caused by the mountain range and adding it to the background field. But since all we want to calculate is the difference between the force and the expected force (the anomaly), we can just calculate the force caused by the mountain range. We also have to include the fact that dense material has been displaced beneath the mountain range, so we have to include a negative force due to the absence of this material. In the simplest case, let's measure the force from a helicopter hover at altitude a above center of the mountain range. From the symmetry of the problem we again know that the force will be only dawnward, and it should have magnitude

$$\delta F = -Gm_0 \int_{-l/2}^{+l/2} dx \left[\int_{-d}^{h} \rho_1 dz - \int_{-d}^{0} \rho_0 dz \right] \int_{-l/2}^{+l/2} dy \frac{\cos\alpha}{s^2}.$$

where $s^2 = x^2 + y^2 + (a-z)^2$. In this case, $\cos\alpha$ (which we define in the same way as we did for the case of the earth above) is equal to $(a-z)/s$, so the force becomes

$$\delta F = -Gm_0 \int_{-l/2}^{+l/2} dx \left[\int_{-d}^{h} \rho_1 dz - \int_{-d}^{0} \rho_0 dz \right] \int_{-l/2}^{+l/2} dy \frac{a-z}{(x^2+y^2+(a-z)^2)^{3/2}}$$

A judicious use of *Mathematica* shows us that the answer is (wait, wait, wait, *Mathematica* is not always fast) not integratable. So we instead turn to numeric methods. A numeric solution the problem, done simply by breaking the mountain up into small squares and calculating the force caused by each, is shown in Figure 9.3. In

So we can see from this plot that the magnitude of the gravitational anomaly *decreases* with the depth of the isostatically compensated region. This result makes intuitive sense; dense material is being replaced by less dense material, so the force caused by this "negative mass" slowly begins compensating for the extra mass of the mountain above. The growth is slow because the distance keeps increasing, too. Eventually, though, there is so much isostatic compensation that the sign of the anomaly flips and there is a negative anomaly (i.e. the total gravitational field is *less* than average).

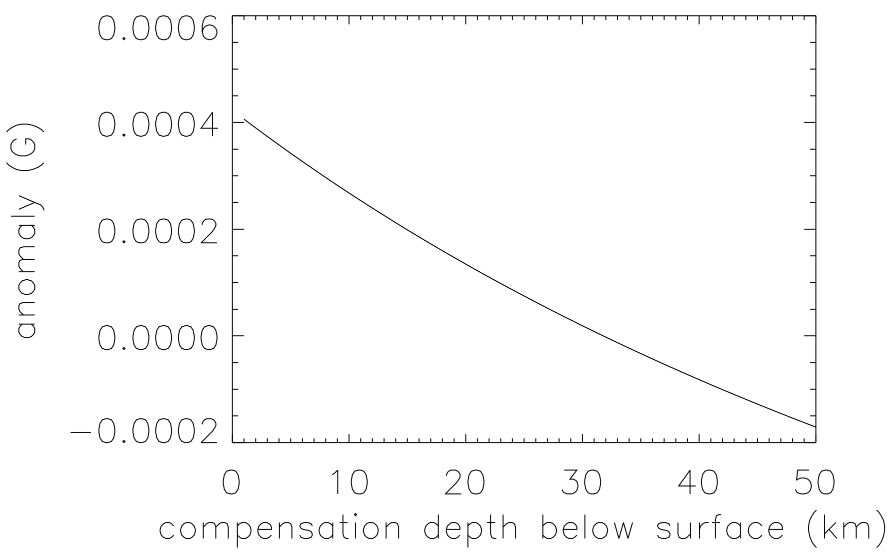

Figure 9.3: The gravitational anomaly caused by a 100 km square mountain of 20 km height for differing depths of the mountain, in units of the earth's gravity.

By measuring the gravitational anomaly, then, we can then make the fundamental geophysical measurment to see what mountain roots look like. What is the answer? To good approximations, the light mountain crustal material floats on the heavier mantle material. *Archimedes principle* states that for an object to float it must displace a mass of liquid equal to its own mass. Thus the mass of material that would have existed in the compensating region must be equal to the total mass of the mountain (including the compensating mass), so

$$\rho_0(d+h)l^2 = \rho_1 d l^2$$

or

$$d = \frac{\rho_0 h}{\rho_1 - \rho_0}.$$

If, for example, the mantle density is twice that of the crust, a 20 km high mountain must have a 10 km deep root to float it.