Lecture 6

Radiative Transfer

6.1 Kirchoff's law

Saying that the earth has an albedo of 39% is a better approximation to reality than saying that it is a blackbody, but even this is not true. The real earth (even averaged over all locations) has a color to it. Albedo is a function of wavelength. Objects that appear red (Mars, Io, Trojan asteroids) have higher albedos in the red part of the spectrum than the blue part of the spectrum, while objects like Uranus appear very blue because of the high blue albedo.

Pondering this fact leads us to the somewhat odd conclusion that things are actually opposite to the colors that they actually appear. A leaf is green because is absorbs all of the light except green. So, in a sense, a leaf is everything *but* green.

Now let's consider some objects that has an albedo that is a function of wavelength that is given by some arbitrary function $A(\lambda)$, and let's place the object in a radiation field which could be arbitrary but, for convenience, we will specify as a blackbody, $B(\lambda, T_{\text{absorb}})$ (Figure 6.1). At each wavelength the body absorbs an amount of radiation $(1 - A(\lambda))B(\lambda, T)$. The body then heats up until it is in thermal equilibrium. What is the spectrum of emission? Let's call the emission efficiency at each wavelength $\kappa(\lambda)$. The emitted radiation is then $\kappa(\lambda)B(\lambda, T_{\text{emit}})$. *Kirchoff's law* states that, in thermal equilibrium,

$$\kappa(\lambda) = 1 - A(\lambda).$$

This law is more profound than it appears at first glance. Rather than saying simply that the amount of energy emitted equals the amount of energy absorbed, it says that the efficiency of emission equals the efficiency of absorption *at each wavelength*. Let's examine what would happen if this were *not* true. Go back to Figure 6.1 and think what would happen if the absorbed power were as plotted in the Figure and the emitted power were simply a blackbody with total

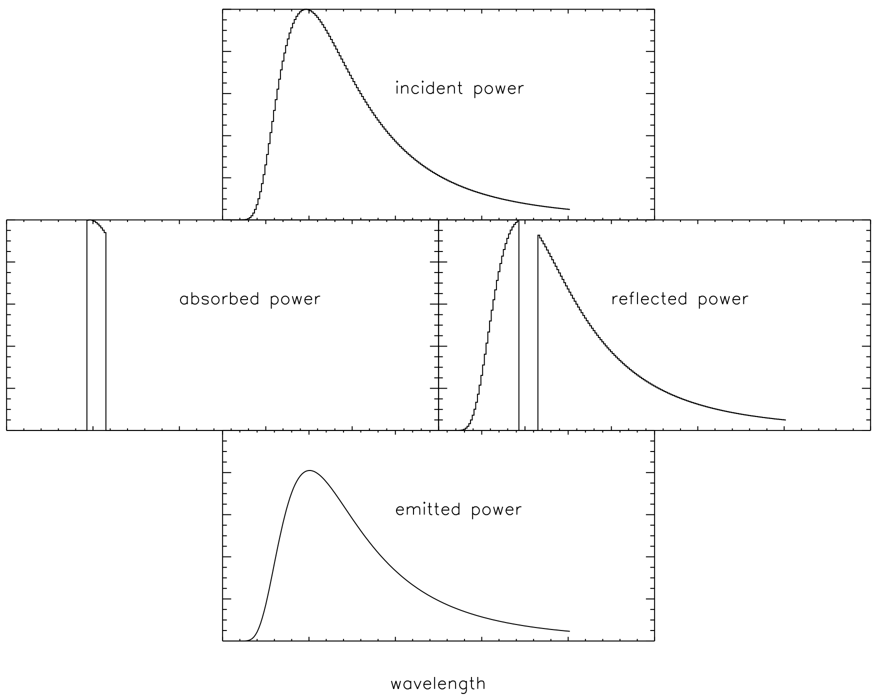

Figure 6.1: An example of a non-blackbody. The incident spectrum is in the shape of a blackbody spectrum, but the albedo of the object is such that it reflectes perfectly except for a small range of wavelengths at which it absorbs perfectly. In the absence of Kirchoff's law of emissivity, the outgoing radiation would be a blackbody with a temperature such that the total amount of energy radiated was equal to the energy absorbed in the narrow wavelength range.

power equal to the absrobed power, as in the Figure. This state would violate Kirchoff's law, which claims that since the object is incapable of absorbing at most wavelengths, it is also incapbable of emitting at those wavelenghts. But is this law true? We'll try to prove it by showing that if it were not true the universe would not operate as expected. In the example of the Figure in which the emitted power is a blackbody spectrum, consider placing a filter between the incident power and the object. This filter perfectly transmits all light at the wavelength that the object absorbs, and perfectly reflects at all other wavelenght. The power absorbed by the object is exactly the same as before, since all of the relevent radiation gets through, but what about the emitted power? If the emitted power were a blackbody as drawn, most of it would hit the filter and be reflected back towards the object. But the object also reflects at these wavelengths, so it would reflect back again. The radiation would be trapped and would never escape. The amount of power trapped between the object and the filter would eventually reach infinity, which really doesn't make much sense. How can we escape this situation? Only if the object only radiates at the wavelengths at which it absorbs, which is approximately a restatement of Kirchoff's law.

Again, this statement is not as obvious as it initially appears. A rose is red because its albedo is such that it absorbs blue light and reflects red light. If the rose were heated to a few thousand degrees Kelvin and were somehow magically prevented from melting, it would be incapable of emitting in the red region of the spectrum, and instead all of the light would come out in the *blue!*. So what color *is* a rose, really?

6.2 Radiative transfer

We now know most everything that we need to know about radiation to derive the equations of *radiative transfer*. Radiative transfer simply means keeping track of radiative and its absorption and emission as it travels along its path. The equation of radiative transfer will turn out to be conceptually quite simple but, in practice, almost impossible to solve analytically. Almost everyone who does any type of radiative transfer for a living solves the equation on a computer. Conceptually, these programs to solve these equations are straightforward, though in practice they can be quite complex. We'll explore some simple versions of such codes.

Consider a point inside a medium (a gas, a solid, a liquid, whatever) which has a temperature $T(\mathbf{x})$ at that point (where by $\mathbf{x}$ we mean that x is a vector and the temperature is known in three dimensions. In general, radiation is flowing through the medium. We'll call the intensity of the radiation I_ν, where I_ν is a function of the position $\mathbf{x}$ and also of the frequency of the radiation, ν. (Note that we are now changing from considering *wavelengths* to *frequencies*. No big deal: we know that $\nu = c/\lambda$. But it's important to know which way you are working the problem, thus the ν subscript will be used whenever we are working with frequencies, while the λ subscript will be used for wavelengths). The units of I_ν are $\mathrm{W/m^2}$/hertz/steradian. The interpretation is a power density ($\mathrm{W/m^2}$) per unit frequency range (1/hertz) per solid angle (1/sterad). The intensity also has to have a direction of travel, which we will call $\hat{n}$.

If the medium is *not transparent* at ν there is an absorption coefficient at $\mathbf{x}$ given by $k_\nu(\mathbf{x})$ (again, the subsscript ν means we are in frequency units rather than wavelength units), which has units of m^{-1}. As the radiation flows from s to $s + ds$ the absorption will cause of a loss in intensity which is proportional to the magnitude of the intensity:

$$dI_\nu = -k_\nu I_\nu ds.$$

In addition to loss by absorption, the medium is emitting thermal radiation.

This emitted energy is added as a source function. We now know from Kirchoff's law that the efficiency of emission at frequency ν is equal to the frequency of absorption at that frequency, so the source is equal to

$$S_\nu = k_\nu B_\nu(T(\mathbf{x}))ds$$

where we are still in frequency units, so the Plank function is

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1}.$$

Thus, our differential equation for the intensity of radiation becomes

$$\frac{dI_\nu}{ds} + k_\nu I_\nu = k_\nu B_\nu(T(s)).$$

This is our old friend the linear first-order differential equation! We could immediately use equatuoin 1.11 back from the first lecture and find the formal general solution to the equation, which turns out to be

$$\begin{aligned} I_\nu(s_{\text{max}}) = \quad & I_\nu(0)\exp(-\int_0^{s_{\text{max}}} k_\nu(s)\ ds) + \\ & \int_0^{s_{\text{max}}} k_\nu(s)B_\nu(T(s))\exp(-\int_s^{s_{\text{max}}} k_n u(s')\ ds')\ ds \end{aligned}$$

While at first glance this equation may appear to be nothing but a mess, it is, in fact, fairly straightforward to interpret. The first term states that the intensity that exists at positiont $s = 0$ decays exponentially proportional to the absorption coefficient as the ray travels through the medium. The second term shows that the medium itself is a source of more radiation, which depends on the absorption coefficient and hte blackbody function, but also that this emission, too, is attenuated exponentially as it travels through the medium.

As a trivial example of radiative transfer, let's consider the case where we have a ball of gas of length l where $T(s)$ is a constant and k_ν is a constant and the intensity entering on one side of the gas is $I_\nu(0)$. What is the intensity exiting the gas? We can write the general solution for the intensity at any point in the gas as,

$$I_\nu(s) = I_\nu(0)\exp(-k_\nu s) + \int_0^s k_\nu B_\nu \exp(-\int_{s'}^s k_\nu\ ds'')\ ds',$$

or, since we want the emission out the other end of the gas

$$I_\nu(l) = I_\nu(0)\exp(-k_\nu l) + \int_0^l k_\nu B_\nu \exp(-\int_{s'}^l k_\nu\ ds'')\ ds'.$$

We can simplify this to

$$I_\nu(l) = I_\nu(0)\exp(-k_\nu l) + k_\nu B_\nu \int_0^l \exp(k_\nu(s'-l)\ ds'$$

or

$$I_\nu(l) = I_\nu(0)\exp(-k_\nu l) + k_\nu B_\nu \exp(-k_\nu l)\int_0^l \exp(k_\nu s')\, ds'$$

which equals

$$I_\nu(l) = I_\nu(0)\exp(-k_\nu l) - B_\nu \exp(-k_\nu l)(\exp(k_\nu l - 1)$$

or

$$I_\nu(l) = I_\nu(0)\exp(-k_\nu l) + B_\nu(1 - \exp(-k_\nu l)) \tag{6.1}$$

This equation simply says that the initial light is attenuated by the exponential factor $k_\nu l$ and that the emission generated within the gas exponentially approaches that expected from a blackbody.

Let's once again make sure this equation makes physical sense. If l is small (compared to $1/k_\nu$) the first term in the equation becomes simply $I_\nu(0)$, while the second approaches zero. The output intensity is just the input intensity, i.e. the gas is transparaent. This makes perfect sense: if looking through a small amount of gas, we see what is behind it. At the other extreme, as $l \to \infty$, the first term vanishes, while the second becomes just B_ν. This again makes sense: as a gas becomes thicker and thicker you can't see what was behind it and it might as well be a blackbody emitting solid.

We see now that k_ν is the reciprocal e-folding length scale for decay of emission in the gas. If we multiply k_ν by a distance in the gas we have a pure number which tells us how many e-folding lengths we have gone. We call this number the *optical depth.* If the optical depth is large ("optically thick"), light trying to pass through the gas undergoes many e-folding attenuations and is impossible to see. If the optical depth is small ("optically thin", incoming light is hardly attenuated at all.