Problem set 1

Ge 108

October 2, 2008

1 Fun with exponentials

Verify equation (1.8), that

$$\frac{d}{dx}(1/2)^x = \ln(1/2)(1/2)^x$$

2 Random DEs

Solve the following differential equations by using an integrating factor:

$$\text{(a)} \qquad x\frac{dy}{dx} + (1-x)y = xe^x$$

$$\text{(b)} \qquad t\frac{dx}{dt} - kx = t^2 \quad (k \text{ is a constant})$$

For (b), what if $k = 2$?

3 Hydrostatic equilibrium

In the troposphere (first $\sim$ 10 km of the earth's atmosphere), the atmospheric temperature decreases by 6.5 K/km. Assuming a surface temperature of $T_0 = 300K$ and plugging this function for $T(z)$ into the equation of hydrostatic equilibrium, what is the atmospheric pressure on top of Mt. Wilson (elevation $\sim$ 2000 m) when the pressure at sea level is 10^5 Pascal? How much less dense is the atmosphere there than at sea level? What is the scale height at sea level? What is the pressure one scale-height above sea level?

For this problem, you will need to know that the mean molecular mass of air (which is mostly N_2) is 29, and that, in the appropriate units, $R = 8310$ J/kg/K

4 Numeric solutions

On the Ge 108 web page (www.gps.caltech.edu/~mbrown/classes/ge108) click on "problem set # 1" and copy the program "numeric.m" to your computer account.

This program does a numeric solution of the 2-element radioactive decay problem that we solved in class and also computes the exact solution. Run the program for $t_m = 1000$ yrs, $t_n = 100$ yrs, and Δt=1 yr. How accurate is the numeric solution compared to the exact solution?

Now try $\Delta t = 10$, $\Delta t = 50$ yrs, and $\Delta t = 100$ yrs. How accurate are the solutions now?

Go back to the case for $\Delta t = 1$ yr, but now change t_m to 100 yrs. and t_n to 10 yrs. What accuracy do you get?

For this equation, at least, can you come up with a general principle for how small Δt needs to be to get reasonable accuracy for your solution?

5 Making up differential equations

(a) Assume that for a spherical drop of water, the evaporation rate is proportional to the surface area of the drop. Write a differential equation for the radius of the drop as a function of time. What are the units of the proportionality constant that you had to use? Solve the equation!
(b) A lake has a volume of 10^6 m^3 and a surface area of 6×10^4 m^2. Water flows into the lake at an average rate of 0.005 m^3/s. The amount of water that evaporates yearly from the lake is equivalent in volume to the lake's top meter of water. The lake is already full, so it can get no deeper than its current depth. Any additional input of water causes lake water to spill over a dam. Initially, the lakewater is pristine, but at a certain time a soluable, noncodistilling (jargon for "it doesn't evaporate, but it does flow away if the water flows away". Think of, for example, salt) is discharged into the lake at a steady rate of 40 tons/year (1 ton = 10^3 kg). Derive a formula for the concentration of pollutant in the lake as a function of time. How

much pollutant will the lake contain as time approaches infinity? Plot this function, using the real values in MATLAB.