Lecture 1

First-Order Differential Equations

1.1 Population Decay

Consider a collection of one million pennies. Every minute we flip the pennies and remove all of them that come up heads. How many pennies are remaining after a time t? On average, half of the pennies are removed each time, so

$$\begin{aligned}
N(0) &= N_0 \\
N(1) &= N_0 \times (1/2) \\
N(2) &= N_0 \times (1/2) \times (1/2) \\
N(3) &= N_0 \times (1/2)^3 \\
&\cdots \\
N(t) &= N_0 \times (1/2)^t
\end{aligned}$$

To make this equation truly general, we should include a *time constant*, τ, where, in our case, $\tau = 1$ minute, and rewrite as

$$N(t) = N_0 \times (1/2)^{t/\tau} \tag{1.1}$$

(Our million pennies would be all gone in about 20 minutes.) Now, instead, consider a million six-sided dice. These dice are rolled once per minute and all of the dice that come up 6 are removed. The equation for the number of dice remaining is then,

$$N(t) = N_0(5/6)^{t/\tau}.$$

(The dice are all gone in about 76 minutes.)

These two examples share the characteristics that the number of objects removed at any time step is a constant fraction of the number of objects existing. If we write the number of objects removed as ΔN, we have, for each time-step τ

$$\Delta N = f \times N, \tag{1.2}$$

where f is a constant that depends on the speed of the removal (fast for the pennies, slower for the dice, so f for the pennies is larger than that for the dice).

Now, we're going to make a differential equation out of this. As with many things in this class, we are going to make the equation by assuming that we can interpolate from a time step that we know, in this case 1 minute, to an infinitesimally small time step, Δt, and that, for small enough Δt, everything is linear. What do we mean by this? As an example, look at Figure 1.1. The dashed line shows the number of dice remaining after each minute's roll. The solid line shows what happens if we break up each minute into 5 12 second intervals and remove 1/5 of the dice during each period. Take a minute to convince yourself that for this case, with Δt equal to the smaller time interval, that we can approximate equation 1.2 by

$$\Delta N = -f \times N \times \Delta t/\tau,$$

(the minus sign is because ΔN is negative) where ΔN is now the number of dice lost in each smaller time step, Δt.

Rearranging, we get

$$\frac{\Delta N}{\Delta t} = -fN/\tau,$$

or, in the limit that $\Delta \to 0$, we can write this as

$$\frac{dN}{dt} = -fN/\tau, \tag{1.3}$$

which states that the time rate of change of N is equal to some constant fraction of N. As in all equations, it's a good idea to think about the units of each of the variables. N is a pure number (no units, or, if you prefer, units of "dice"), t is a time, and the Δs or differentials only refer to a change in some quantity, so they have no units (i.e., the units of "change in time" are still "time"). For the units on both sides of the equation to be equal, which they must, f must evidently have no units (or units of "dice"), which

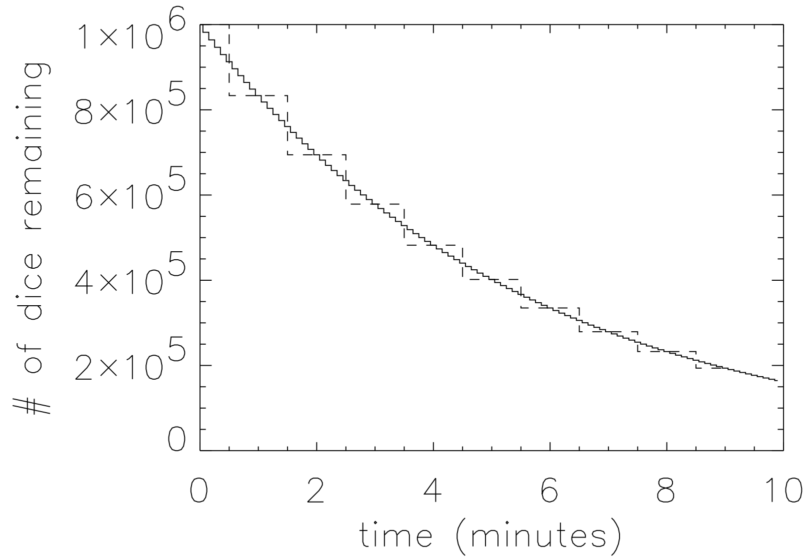

Figure 1.1: The number of dice remaining as a function of time if we start with a million, roll the dice every minute, and remove each die which comes up with a 6. The solid line shows a linear approximation for 12 second time steps.

makes sense because we defined it as a fraction removed, and therefore must be related to the speed with which the decay happens. As we noted above, for fast decay, as for coins, f is larger (so the time scale is shorter), while for slower decay, as for dice, f is smaller (so the time scale is longer). An equation like this, with two separate constants, is overly cumbersome, so we are going to redefine $\alpha = f/\tau$ and rewrite our equation more simply as

$$\frac{dN}{dt} = -\alpha N. \tag{1.4}$$

This equation is the *population equation*, and it is our first example of a *first-order differential equation*. The order of a differential equation is simply the highest differential to appear in the equation. Thus

$\frac{d^2x}{dt^2} + t^2 x = \sin t$ is second order,

$$
\begin{array}{ll}
\frac{d^2z}{dt^2} + x\frac{dz}{dx} + z = 0 & \text{is second order,} \\
\frac{dx}{dz} + \sqrt{1-z^2}x = \cos^{-1} z & \text{is first order.}
\end{array}
$$

Most of the important equations in the physical sciences are first or second order differential equations. A basic knowledge of how to set them up and solve them will get you a long ways.

First order DEs are particularly easy to solve, or, at least, to reduce to an integral which, if necessary, is easily evaluated numerically. The population equation is one of the simplest DEs of all. In fact, since we started with the solution to get to the equation we already know more or less that the solution is some form of exponential decay. We'll go through and formally solve the equation anyway just to get some practice with the methods.

The first thing to try with a 1st order DE like equation 1.4 is to integrate both sides of the equation separately. This procedure is just like basic algebra where you multiply both sides of an equation by the same number or variable except that it has an additional complication that we'll see in a minute. To integrate the equation, however, we have to first use the method of *separation of variables* where we isolate the two variables to different sides of the equation by rewriting equation 1.4 as

$$\frac{dN}{N} = -\alpha dt. \tag{1.5}$$

Note that we have treated the differential dN/dt as as a simple variable that can be manipulated like any other. This will be a common practice when working with DEs and you are encouraged to think of differentials this way (as simply $\Delta N/\Delta t$, for example).

From first year calculus, recall that

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

again, playing with the differential

$$d\ln x = \frac{dx}{x},$$

and now integrating,

$$\int d\ln x = \int \frac{dx}{x},$$

becomes

$$\ln x + C = \int \frac{dx}{x}.$$

The constant C is called the *constant of integration*, and it is the same type of constant that you found in 1st year calculus whenever you evaluated an indefinite integral (i.e., $\int x \ dx = x^2/2 + C$). If we replace the variable x everywhere above with the function $f(x)$, we find

$$\ln f(x) + C = \int \frac{df(x)}{f(x)}.$$

Now, going back to equation 1.5, and integrating both sides, we find

$$\ln N + C = -\alpha t.$$

C is again a constant of integration. We will determine its value from the *boundary conditions* of the equation, which, in this case, we will see to be the number of coins or dice or whatever at $t = 0$.

Recall the definition of the natural logarithm – if $\ln x = y$ then $e^y = x$ – and exponentiate both sides

$$e^{\ln N + C} = e^{-\alpha t}$$

or, since $e^{a+b} = e^a e^b$,

$$Ne^C = e^{-\alpha t}$$

and finally, since C is an arbitrary constant, e^C is equally arbitrary, and we will just rewrite the equation as

$$N = ke^{-\alpha t}. \tag{1.6}$$

As before, we look at the units. N is dimensionless, so, therefore, must be k. We still find that α must have units of 1/time.

As a check of our solution to the DE, we can always substitute back into the equation and make sure it works

$$\frac{dN}{dt} = \frac{d}{dt} ke^{-\alpha t} = -\alpha ke^{\alpha t} = -\alpha N.$$

Note that equation 1.6 is like equation 1.1 except now the exponentiating factor is e instead of $1/2$ and the exponent is $-\alpha t$ instead of $-t/\tau$. How are α and τ related? If

$$e^{-\alpha t} = (1/2)^{-t/\tau}$$

then we can take the natural log of both sides to get

$$-\alpha t = -\ln(1/2)^{t/\tau}$$

or, using the property that $\ln a^b = b \ln a$,

$$\alpha t = (t/\tau)\ln(1/2)$$

or

$$\alpha = \ln(1/2)/\tau.$$

The reciprocal of α, which we will call t_e, is called the *e-folding time* of the decay. We can find it's relationship to the half-life by solving for the time at which half of the population is gone: from

$$1/2 = e^{-t_{1/2}/t_e}$$

take the log of both sides to get

$$\ln(1/2) = -t_{1/2}/t_e$$

or

$$t_e = t_{1/2}/\ln 2 \approx 1.44 t_{1/2}, \tag{1.7}$$

that is, the e-folding time is approximately 44% longer than the half-life, which makes sense, because the e-folding time is that time required for $1/e \approx 1/2.718 \approx 0.368$ of the population to decay; slightly more than one-half.

Note that there is no special reason to use half-lives or e-folding times or anything else; as long as equation 1.7 holds, these are completely equivalent equations:

$$\begin{aligned} N &= N_0 e^{t/t_e}, && \text{and} \\ N &= N_0 (1/2)^{t/t_{1/2}}. \end{aligned}$$

We could write the equations using third-lives, tenth-lives, or anything else that we wanted. The only reason that we use e-folding is that it is mathematically simpler, only because

$$\frac{d}{dx}e^x = e^x$$

whereas

$$\frac{d}{dx}(1/2)^x = \ln(1/2)(1/2)^x. \tag{1.8}$$

We would be certain to misplace the factor of $\ln(1/2)$ is we had to carry it around everywhere, so we are going to stick with e^x instead.

1.2 Integrating factors

The population equation is, of course, the basic equation for isotopic decay used throughout isotopic geochemistry. Let's go through a more complicated example than simple radioactive decay of a single element. Consider a system where some radiogenic element N, with an initial population N_0 and an e-folding time of t_n decays to element M, with an initial population of 0 and which is also unstable, and decays with an e-folding time of t_m. What is the differential equation for the amount of M after a certain time? We known the equation for the amount of N:

$$\frac{dN}{dt} = -N/t_n \tag{1.9}$$

What about the equation for M? The equation should have the general form of

$$\textit{time rate of change of } M = \textit{source} - \textit{sink}.$$

We know that the time rate of change of M is simply dM/dt. The source of M is the decay of N; everything lost from N is gained by M, therefore

$$\textit{source of } M = +N/t_n$$

(note that it is positive here, because it is a source, where it was negative in equation 1.9, because it was a sink).

The sink of M is simply the radioactive decay,

$$\textit{sink of } M = -M/t_m.$$

Putting the entire equation together yields:

$$\frac{dM}{dt} = N/t_n - M/t_m,$$

where both N and M are functions of time. We know the solution for N, though, so we substitute (and switch notation: I'm now going to use $\exp x = e^x$ just because it is easier to read in the typesetting),

$$\frac{dM}{dt} = N_0/t_n \exp(-t/t_n) - M/t_m. \tag{1.10}$$

And now we're stuck, because the variables won't separate and the method we used above won't work. We'll have to use a trick (for which we will derive the general form in a minute) of using an *integrating factor*. There is no reason that you should be able to have guessed this yourself, so just hang on for the ride for a minute.

First, let's multiply all of equation 1.10 by $\exp(t/t_m)$. We now have

$$\exp(t/t_m)\frac{dM}{dt} + \exp(t/t_m)M/t_m = N_0/t_n \exp(-t/t_n + t/t_m). \tag{1.11}$$

Note now that we can use the chain rule to find

$$\frac{d}{dt}(M\exp(t/t_m)) = \frac{dM}{dt}\exp(t/t_m) + \frac{1}{t_m}M\exp(t/t_m),$$

which shows us we can rewrite equation 1.11 as

$$\frac{d}{dt}(M\exp(t/t_m)) = N_0/t_n \exp(-t/t_n + t/t_m).$$

If we integrate both sides of this equation, we have

$$M\exp(t/t_m) = N_0/t_n \int \exp(-t/t_n + t/t_m)\, dt + C$$

or

$$M\exp(t/t_m) = N_0 \frac{t_m}{t_n - t_m}\exp(-t/t_n + t/t_m) + C.$$

Finally, dividing through by $\exp(t/t_m)$, we have the solution:

$$M = N_0 \frac{t_m}{t_n - t_m}\exp(-t/t_n) + C\exp(-t/tm).$$

The constant C must be chosen to satisfy our initial condition that $M(t = 0) = 0$. Evaluating the expression, we find

$$M(t = 0) = N_0 \frac{t_m}{t_n - t_m} + C,$$

therefore

$$C = -N_0 \frac{t_m}{t_n - t_m},$$

and our equation is solved as

$$M = N_0 \frac{t_m}{t_n - t_m}[\exp(-t/t_n) - \exp(-t/t_m)]. \tag{1.12}$$

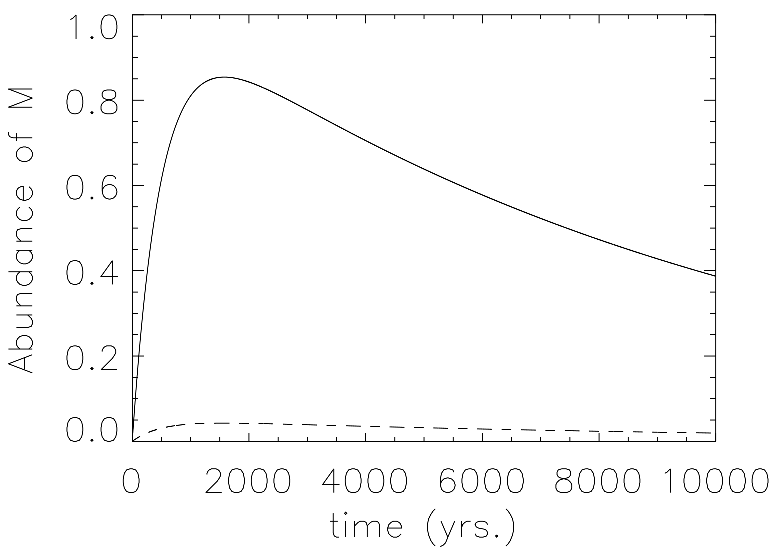

Figure 1.2: The abundance of radiogenic daughter species M. The solid curve shows the abundance for a parent e-folding time of 500 yr and a daughter e-folding decay time of 10000 yr The dashed curve shows the abundance for 10000 and 500 yr parent and daughter e-folding times, respectively.

Figure 1.2 shows the solution for two value of t_n and t_m.

As an aside, whenever you solve any equation, check to see if the answer makes sense. One way to do this is to see what happens at different large and small limits of either the variables or the parameters. In the above problem, we can intuit the general form of the solution in the limits when either $t_n \gg t_m$ or when $t_n \ll t_m$. In the former case the decay of N is slow and the decay of M is fast, so the decay products of N which form M disappear quickly. The solution should approach zero. Evaluating equation 1.12 for $t_n \gg t_m$, we see that the factor $t_m/(t_n - t_m)$ in front will be small and the solution will tend toward zero, as expected.

For the case where $t_n \ll t_m$, the solution should approach that of the case where all of N has turned to M and M is decaying. Here, the factor $t_m/(t_n - t_m)$ approaches -1 and $\exp(-t/t_n)$ approaches zero much faster than $\exp(-t/t_m)$, so the solution simply approaches $\exp(-t/t_m)$, as expected.

One other thing to watch out for in solutions is places where there might be problems. For example, in equation 1.12, what happens if $t_n = t_m$? Physically, nothing singular should happen at this point, but the algebraic factor at the beginning of the equation become 1/0! Luckily, the factor in brackets becomes zero, also, so we have a 0/0. Remembering L'Hopital's rule from calculus, we could show that, indeed, this solution is OK.

1.3 The General Solution

The method of using an integrating factor that we used in the previous example is extremely powerful and can be used to solve *any* 1st order *linear* differential equation. We've already defined 1st order DEs to be those with only first derivatives. *Linear* DEs are those that contain only linear terms in the dependent variable (M in the example above). Note that the *independent* variable can do anything it wants (though it may make the solution harder). Carefully consider the following examples:

$\frac{d^2x}{dt^2} + kx = 0$	linear, second order
$\frac{dx}{dt} + t^{5/2}x = e^t$	linear, first order
$\frac{d^2x}{dt^2} + k(\frac{dx}{dt})^2 + \alpha x = 0$	non-linear, second order
$\frac{dx}{dt} + 3/x = 0$	non-linear, first order.

Many problems in the earth sciences can be solved with linear first-order DEs. *Any* of these problems can be solved with the following method. This method will not be something general that can be applied to other sorts of DEs, but, unfortunately, that is the way with DEs. The real trick is to know how to reduce the problem or equation that you have into something you know how to solve. And if the equation is a linear 1st order DE, you know that it can be solved with this method.

All 1st order linear DEs can be rewritten as

$$\frac{dx}{dt} + p(t)x = f(t)$$

(in our example above, $p(t) = 1/t_m$ and $f(t) = N_0/t_n \exp(-t/t_n)$). Like before we now multiply through by an integrating factor equal to $\exp(\int p(t')\,dt')$, giving

$$\exp(\int p(t')\ dt')\frac{dx}{dt} + \exp(\int p(t')\ dt')p(t)x = \exp(\int p(t')\ dt')f(t).$$

We now notice that

$$\frac{d}{dt}x(\exp(\int p(t')\ dt')) = x\frac{d}{dt}(\exp(\int p(t')\ dt')) + \frac{dx}{dt}\exp(\int p(t')\ dt')$$

but since

$$\frac{d}{dt}(\exp(\int p(t')\ dt')) = \exp(\int p(t')\ dt')\frac{d}{dt}\int p(t')\ dt' = \exp(\int p(t')\ dt')p(t),$$

we can write

$$\frac{d}{dt}(x\exp(\int p(t')\ dt')) = xp(t)\exp(\int p(t')\ dt') + \frac{dx}{dt}\exp(\int p(t')\ dt')$$

and simplify to

$$\frac{d}{dt}(x\exp(\int p(t')\ dt')) = f(t)\exp(\int p(t')\ dt')$$

This equation can be integrated on both sides to give

$$x\exp(\int p(t')\ dt') = \int f(t)\exp(\int p(t')\ dt')\ dt + C$$

and now dividing both sides by $\exp(\int p(t')\ dt')$ we find the *formal general solution*

$$x(t) = \exp(-\int p(t')\ dt')\int f(t)\exp(\int p(t')\ dt')\ dt + C\exp(-\int p(t')\ dt') \tag{1.13}$$

for the 1st order linear differential equation:

$$\frac{dx}{dt} + p(t)x = f(t).$$

This may not look like much of a solution, but in many cases the integrals are exactly solvable (as in the population equations above), and even if they are not, many good numeric integration packages are available these days that will spit out a quick solution for you.

Let's try a few examples that have no particular physical basis but will be good just to see how the math works out. Consider the equation:

$$\frac{dx}{dt} + 3x = t.$$

Here, p(t)=3 and f(t)=t, so we can use equation 1.13 to simply plug in and write:

$$x(t) = \exp(-3t) \int t \exp(3t)\ dt + C \exp(-3t)$$

and remembering that

$$\int z \exp(az)\ dz = \frac{az - 1}{a^2} \exp(az)$$

we have

$$x(t) = \frac{3t - 1}{9} + C \exp(-3t).$$

Like before, the constant of integration, C, needs to be determined from the initials conditions, such as $x(t = 0) = 33$, or any other single specified value of x at any time.

Again, though this is a one-off mathematical trick that you won't apply elsewhere, 1st order linear DEs pop up in many surprising places, so being able to recognize them and remember how to solve them is an important skill. You'll get some practice in this both in the problem set and in the next lecture, when we consider some physical examples.