12 Gravity and Topography

In the limit of perfect hydrostatic equilibrium, there is an exact relationship between gravity and topography.... and no new information emerges: The topography is directly determined from measuring the gravity and vice-versa. (The surface is an equipotential).

In the limit where the only non-uniformity in the body is topography (i.e. a uniform density body that is “bumpy” with the bumps supported by finite strength), there is also an exact relationship between gravity and topography because the topography tells us what gravity anomalies to expect. No new information emerges (except the confirmation that the gravity anomalies are indeed merely the consequence of the topography alone and the density of the material is uniform). One could, however, determine the density of the near surface material (i.e., crust) because the gravity directly associated with the topography depends on this.

In the real situation of a solid planet, there is new information available from studying gravity and topography together because the density anomalies responsible for the gravity are neither entirely hydrostatic nor entirely confined to the surface and expressed as topography. However, this real situation is also very non-unique in its interpretation because the depth of the density anomalies is undetermined (unless you can use yet another data set, e.g., seismic tomography). The non-uniqueness problem is reduced by appealing to physically plausible models: ideas of isostasy, the nature of the lithosphere, and ideas about the nature of the dynamics resulting from density anomalies in a viscous mantle. To some extent, these ideas are testable by looking at the spectra, the nature of gravity and topography (and their interrelationship) in spherical harmonic space. This is a big subject and this chapter only summarizes the main points.

12.1 Uncompensated Topography

Suppose that there is a topographic harmonic $H_{lm}$ in the crust (density $\rho_c$), so that the resulting mass anomaly per unit area is $\rho_c H_{lm} P_{\ell m}^m (\cos \theta) \cos \phi$. The contribution to the gravity potential external to the planet is then

$$\Delta V_{lm} (r, \theta, \phi) = GP_{\ell m}^m (\cos \theta) \cos m \phi \int_{-1}^{1} d \cos \theta' \int_0^{2\pi} d \phi' \int_0^{R + H_{lm} P_{\ell m}^m} \rho_c r'^2 (r'/r + 1) P_{\ell m}^m (\cos \theta') \cos m \phi' dr'$$

$$= GP_{\ell m}^m (\cos \theta) \cos m \phi \frac{4 \pi R^2 H_{lm} \rho_c}{2 \ell + 1} \frac{R^\ell}{r'^{\ell+1}}$$

and this can be written in the form:

$$\frac{3GM \rho_c}{\bar{\rho}(2\ell + 1)} \frac{H_{lm} R^\ell}{R \cdot r'^{\ell+1}}$$

(12.1)

where $\bar{\rho}$ is the mean planet density and the Legendre function, etc. has been dropped for simplicity of writing (i.e., it is always implied.)
12.2 Compensated Topography

Suppose we wished to “compensate” for this topography at some depth D. Compensation in the isostatic sense means that the total mass at any location above depth D must be the same as it is in a location where there are no anomalies whatsoever.

In other words, if there is a density anomaly $\Delta \rho$ at depth D then it must have a thickness $H_{im}^*$ such that

\[
(R - D)^2 H_{im}^* \Delta \rho = -R^2 H_{im} \rho_c
\]  

This is what we mean by isostatic compensation in a spherical planet. See the figure below. [But note that there are other kinds of definitions that one could construct for “isostasy”. Ultimately one wants a specific physical model]. The parameter D is called the depth of compensation. It is a theoretical construct and might or might not correctly describe what happens in a real planet. Of course, positive topography must be compensated by a negative density anomaly (i.e., a density deficit). This could, for example, be a crustal root, but it could also be a hotter part of the mantle.

\[\text{Figure 12.1}\]

It should be evident by inspection of the integration above that the gravity potential anomaly associated with the density anomaly at depth D is:
\[ \Delta V_{\ell m} = -\frac{3GM\rho_c}{\bar{\rho}(2\ell + 1)} \cdot \frac{H_{\ell m}}{R} \cdot \frac{(R - D)^\ell}{r^{\ell+1}} \quad (12.3) \]

from which it follows that the total gravity potential anomaly is now:

\[ \Delta V_{\ell m}(\text{total}) = \frac{3GM\rho_c}{\bar{\rho}(2\ell + 1)} \cdot \frac{H_{\ell m}}{R} \cdot \frac{R' - (R - D)^\ell}{r^{\ell+1}} \quad (12.4) \]

### 12.3 Geoid and Its Relationship with Topography

The geoid height is defined to be the distance above a reference surface (e.g. mean radius of planet, including the effect of rotation) of a constant potential surface. So if we define the height for a given harmonic component \( G \) then by definition

\[ \frac{GM}{R + G_{\ell m}} + \Delta V_{\ell m} \quad (12.5) \]

is independent of position (i.e. is a constant). [Remember that the .... after \( G_{\ell m} \) etc means the usual Legendre function or spherical harmonic]. So it follows (using first order Taylor series expansion) that

\[ G_{\ell m} = \frac{3\rho_c H_{\ell m}}{\bar{\rho}(2\ell + 1)} \quad \text{[No compensation]} \]

\[ G_{\ell m} = \frac{3\rho_c H_{\ell m}}{\bar{\rho}(2\ell + 1)} (1 - \zeta') \quad \text{[Includes isostasy]} \quad (12.6) \]

\[ \zeta \equiv 1 - \frac{D}{R} \]

For \( D/R \ll 1 \) and \( \ell \) not too large, \( 1 - \zeta' \approx \ell D/R \) and \( G \) is only weakly dependent on \( \ell \) and proportional to \( D \). By comparing this prediction with data, one might hope to deduce the depth of compensation \( D \).
Figure 12.2

Shown above is an attempted application to Mars. As you can see, it does not work very well. However, it does show that compensation (in some sense) is occurring and it requires relatively small values of D, e.g. 200km or less.

One reason why the data are fit rather poorly is mantle dynamics, which is important at small harmonic degree (see below). Another reason is that the lithospheric flexes in a way that depends on some other length scale (related to but not equal to the depth of compensation D). This flexural response provides compensation for long wavelengths but poor compensation at shorter wavelengths (high harmonics). Flexure is discussed in an appendix to this chapter. Another reason is that the planet does not have uniform properties from region to region (i.e., the parameter D is not constant with latitude and longitude).

One way to see the extent of isostasy is to look at the power spectrum. This adds up the sum of the squares of harmonics for a particular \( \ell \) value. Below, this is plotted in such a way that the topography and geoid power spectra would exactly agree if there was no compensation. The much lower variance of the geoid (at least at low harmonic degree) attests to the extent of isostasy.
Figure 12.3

In the case of Mars, most of the topography is associated with variations in crustal thickness and is very well compensated. This is illustrated below, showing the results of the MGS mission (Zuber et al, *Nature*, 2001) and the inferences for crustal thickness. Essentially, one uses the gravity field to infer that the topography is primarily hydrostatic and then constructs a gravity thickness model that agrees with the observed topography.

Note, however, that there is non-uniqueness because we don’t know the crustal density exactly and we need a “calibration point somewhere – a place where we know the crustal thickness by other means. It is not clear whether we can claim to have that well determined, though you certainly cannot make the crust very thin (else the mantle would “poke” through somewhere, and it does not). And you cannot make the crust too thick (it would tend to flow sideways). The details of the gravity can in principle reduce the uncertainties.

Fig. 15. Variance spectra of gravity and equivalent gravity from topography for terrestrial planets and the Moon.
12.4 Role of Dynamics

Especially at low harmonic degree, we expect most of the gravity and at least part of the topography to arise as shown in the cartoon below.
Generally (as for isostasy above but more broadly applicable), we expect:

\[ G_{\text{in}} = F_{\text{in}} H_{\text{in}} + E_{\text{in}} \]  

(12.7)

where the F tells you something about the planetary structure and the E is an error or noise term. The G can be thought of as having two pieces (at least): The direct effect on gravity due to the density anomaly and the dynamic topography effect arising from the viscous flow driven by that anomaly. These are opposite in sign and the latter has a magnitude that depends on the viscosity structure. As a result one cannot predict even the sign of the total geoid effect with confidence. Inevitably, it follows that the interpretation is highly non-unique. (On Earth, this non-uniqueness is removed by appealing to seismic data.)

The F is known as a spectral impedance. This kind of modeling turns out to depend on the lithospheric thickness (because this restricts the density anomalies to greater depth... high lithospheric thickness generally leads to larger geoid anomalies somewhat in analogy with the simple isostasy calculation above). It also depends on vigor and style of convection. And very importantly it depends on the viscosity structure of the mantle and the existence of any density interfaces or phase changes.
12.5 Appendix: Flexure of an Elastic Lithosphere

When you load the outer elastic part of a planet, it flexes. This deformation can be observed and also expresses itself in the gravity field to an extent that depends on whether the flexing compensates for the load.

Consider a plate bent as shown. It has a radius of curvature, R. By Pythagoras,
\[(R - \delta)^2 + \left(\frac{\lambda}{2}\right)^2 = R^2\]
\[\Rightarrow R \sim \frac{\lambda^2}{8\delta}\] (because $\delta \ll R$) \hfill (12.8)

Now the upper surface of the plate is shorter than the mid ("neutral membrane") surface of the plate by $(R - d/2)\theta - R\theta = -d\theta/2 \sim -d\lambda/2R$. So the strain in the plate varies linearly from $-d/2R$ at top surface (compressive) to $d/2R$ (extension) at the bottom surface. The elastic energy at any location is $\sim Ec^2/2$ where E is Young’s modulus and $c$ is the local strain. So the total elastic energy stored per unit area of plate surface is $\sim \frac{1}{2}E(d/2R)^2d/3$ where the factor of three comes from integration from top to bottom. Substituting $R \sim \frac{\lambda^2}{8\delta}$ gives $8Ed^3\delta^2/3\lambda^4$. 

Figure 12.6

Figure 12.7
The force required (per unit underside area) to create the displacement $\delta$ is simply the derivative of energy with respect to displacement (that’s the definition of the relationship between force and work done!) This is $F = 16Ed^3/3\lambda^4$. (It has units of pressure of course).

Notice three things about this:

(i) It depends on $Ed^3$. It turns out to be useful to characterize the plate’s stiffness by a property called flexural rigidity $D=Ed^3/12(1-\nu^2)$ where $\nu$ is Poisson’s ratio. (The above crude derivation does not take this ratio into account). Clearly, $Ed^3$ is the crucial thing.

(ii) The force is linear in vertical displacement just like Hooke’s law, so the behavior is linear.

(iii) The force varies inversely as the fourth power of the lateral length scale describing the system. This suggests that the correct differential equation for the system involves the fourth derivative of the displacement with respect to horizontal coordinate. And indeed that is so.

12.5.1 What is the preferred Wavelength for a Point Load?

The following is a heuristic argument. Suppose we load the plate with a point mass $M$. This will be balanced over an area $\sim \lambda^2$ by the elastic response (above) and by the displacement of mantle fluid (density $\rho_m$):

![Diagram showing point load on a plate](image)

Fluid pushes up force/area $= \rho_mg\delta$; plate pushes up with force/area as in text

Figure 12.8

(This is a cross-sectional view; think of the region affected as a circle of radius $\sim \lambda$). So $Mg = (\rho_mg\delta + 16Ed^3/3\lambda^4) \lambda^2$.

The choice of $\lambda$ that the system will make is the one that maximizes $\delta$. The reason is that this minimizes the total energy (i.e., maximizes the gravitational energy release, which is $Mg\delta$). Notice that if you make $\lambda$ very large then that is bad because it makes the work done in displacing mantle fluid large. But if you make $\lambda$ small then that makes the elastic energy very large. The optimum choice comes from taking the derivative with respect to $\lambda$ and setting it to zero:

$$d/d\lambda[(\rho_mg\delta + 16Ed^3/3\lambda^4) \lambda^2]=0 \Rightarrow \lambda_c = [16Ed^3/3\rho_mg]^{1/4} \tag{12.9}$$

This the characteristic wavelength on which the plate will bend. For Earth, $E \sim 10^{11}$ Pa, $d\sim30$ km, $\rho_m \sim 3000$ kg/m$^3$, $g = 10$ m/s$^2$, we get $\lambda_c \sim 10^{5.2} \sim 200$ km. If you measure $\lambda_c$ then this can be used to determine $d$, the thickness of the plate. This has been applied to Mars, Venus, Europa, Ganymede, etc.
We can now ask: How is the isostatic approximation modified? Suppose we place a load of crust of thickness $H$ on the lithosphere and there is a downward displacement $\delta$. At the plate we have the following force balance:

$$\rho_c g H \sim 16E d^3 \delta / 3 \lambda^4 + \rho_m g \delta$$  \hspace{1cm} (12.10)

representing the load acting down on the plate, the upward elastic force of the plate and the upward mantle force respectively. We’re assuming a horizontal lengthscale $\lambda$ that is larger than $\lambda_c$. Then from the equation above for $\lambda_c$, the elastic term is smaller than the mantle displacement term by $(\lambda_c/\lambda)^4$ and the solution approaches the isostatic solution $\rho_c g H \sim \rho_m g \delta$. Important conclusion: The system approaches isostatic response when the load is over a lengthscales larger than a few hundred km. This is why continents are very close to isostatic whereas the island of Oahu (for example) is not.

**Ch. 12 Problems**

12.1) (a) Explain why the depth of compensation $D$ is plausibly larger on small planet than on a large planet, other things being equal. To do this you should to think about two different ideas for depth of compensation: (i) That it is roughly the depth at which nearly typical mantle temperatures (e.g. 1200-1500K) are reached. Why? (ii) That it is related to the thickness of the crust and thus is related to the depth range over which melts can be produced. This will be determined by the pressure in a planet (why?) and you have to go deeper to reach a particular pressure on a small planet. Can you suggest a simple rule for how $D$ might scale with $R$ (the planet radius), assuming all other factors are equal between the two planets?

(b) We found that for a simple isostatically compensated model, the geoid to topography ratio ($G_{lm}/H_{lm}$) is $3\rho_c (1 - \zeta)/(2\ell + 1) \rho_{av}$ where the symbols are as defined in this chapter (with $\rho_{av} =$ mean planet density). For features in topography (and of course geoid) with lateral extent (i.e., wavelengths) of order a few hundred km, estimate how the geoid to topography ratio should scale with planet size.

[Of course, you need to relate $\ell$ to wavelength. You might wish to present your result as a sketched graph; don’t go into anything mathematically complicated. You may wish to use a Taylor series expansion somewhere, but take care to check whether it is legitimate.]

12.2) (a) Cassini gravity data for Titan suggest a non-hydrostatic degree 2 geoid anomaly of perhaps $3 \times 10^{-6}$ (in units where “unity” would correspond to the degree 0 gravity $GM/R$). You can think of this as the coefficient in front of $P_2$ (a degree 2 Legendre function where the axis of that function is arbitrary). One way to do this is the “iceberg effect” in which Titan has an ocean that begins at depth $D$ (~100 to 200km) below the surface, so the geoid anomaly comes from the bit of ice that sticks above the surface and the isostatically compensating “keel” of ice that sticks down into the ocean. (Both of these have thickness small compared to $D$). As a function of $D$, what is the required topographic relief on
the surface, peak to peak? (“Peak to peak” means from the minimum of the $P_2$ function to its maximum. Assume the ice has density 0.92 g/cc and the ocean has density of 1.0 g/cc. The mass of Titan is $1.35 \times 10^{26}$ g and the radius of Titan is 2575km.

(b) Suppose instead that this anomaly is from non-compensated topography (which means it would be too small to be detected.) What is the necessary thickness of the elastic lithosphere on Titan required for this to be plausible? This thickness must in reality be much less than the depth to the ocean, D above. Why? Assume a Young’s modulus for ice of $10^{11}$ dynes/cm$^2$. (This is quite uncertain but your answer is only weakly dependent on this choice.) For this part you need to read and understand the appendix of ch 12 and all I’m asking you to do is apply the formula for the characteristic wavelength of deformation for a load (and to understand how it should be applied.)