1 Problem 3.3

2. Problem 4.6 (not in the current text) Here is a crude model for the molecular-metallic hydrogen phase transition (at T=0). Relative to infinitely dispersed hydrogen molecules, the energy of molecular hydrogen is roughly $E_1(V)$ and the energy of metallic hydrogen is roughly $E_2(V)$, where

$$E_1(V) = 0.15/V^3$$
$$E_2(V) = 0.1 + 0.05(V/1.6 - 1)^2$$

(The first represents the strongly repulsive interaction of hydrogen molecules and ignores the tiny attractive part at large V. The second is just the beginning of a Taylor series expansion about the zero pressure V of 1.6; the linear term must of course be zero at zero pressure. Here the units are Rydbergs/electron for energy where V is in units of cm$^3$/gram, the reciprocal of mass density).

Find the pressure of the phase transition and the densities of the co-existing phases at the transition. Best to do this graphically (Mathematica can do this for you in a few lines) because you can then see what is happening (though of course Mathematica can also do it just be solving a set of non-linear simultaneous equations). In getting the pressure, be careful with units since you want the answer in standard units (e.g., cgs or Megabars or whatever). Remember that 1 gram of hydrogen contains Avogadro’s number of electrons.

3. Problem 5.1

4. Problem 5.4 (not in the current text).

From a combination of Kepler data and Doppler (or Transit time variations), we now have a lot of planets for which we have both radius and mass. A striking feature of these data (for mass $<\sim 10$ Earth masses) is for the tendency of the radius to increase more rapidly with mass than the expectation for any single fixed composition condensed material (i.e., more rapidly than $M^{1/3}$ or even $M^{1/3}$.) This is because the more massive bodies must have captured gas and these puffy envelopes greatly increase the radius. The purpose of this problem is to find an approximate model that describes this. Consider a planet that is mostly rocky but with a low mass envelope. The rocky core has mass $M_r$ and radius $R_r$. Because the envelope is assumed to have low mass, we can assume that the gravity is $GM_e/r^2$ throughout that envelope. We further assume that the envelope has two parts, an essentially isothermal outer part and an isentropic (convective) inner part. The reason for this is that the energy coming from the core of the planet is small compared to the absorbed starlight high up in the atmosphere. The isothermal part has negligible mass and extends from a pressure of 10 millibars (this is defined by optical depth unity for transit observations) to, say, 100 bars (that’s many scaleheights!) The pressure in the isentropic part is $p=p_{i1}(p/p_{i1})^{1/4}$ where $p_i$ and $p_{i1}$ are the pressure and density at the beginning of the convective zone. Assume the envelope is pure molecular hydrogen and the effective temperature of the planet (also the temperature throughout the radiative zone, roughly) is 500K.

(a) Show that the density in the convective zone $p/p_{i1} = [1 + A(1/x - 1/x1)]^{1/2}$, where $A=2GM_e/7c_e^3r_{i1}$; $x=r/R$; $x1 = R_1/R_r$; $c_e^2=p_{i1}/p_{i1}$ and the radius $R_1$ is the place where $p/p_{i1}=1$. (Note that $A$ is related to something of obvious physical significance: The extent to which the gas is gravitationally bound).

(b) Confirm that the mass of the envelope is approximately $M_e=4\pi \rho_1 R_e^2 f(A)$ where

$$f(A) = \int_1^{x1} \left[ 1 + A \left( \frac{1}{x} - \frac{1}{x1} \right) \right]^{5/2} x^2 \, dx$$
Calculate \( f \) numerically (Mathematica will do this in one line) and hence estimate the value of \( x_1 \) for which the mass of the envelope is 1\% of the mass of the planet; assume \( M_c = 5 \) Earth masses, \( R_c = 1.6 \) Earth radii.

(c) Confirm that the outer radius \( R_o \) is given by solving \( 10^4 = \exp\left[3.5A(1/x_1 - 1/x_2)\right] \) where \( x_2 = R_o/R_c \). Hence (one line in Mathematica) determine the outer radius of the planet.