Numerical technique for solving the radiative transfer equation for a spherical shell atmosphere

B. M. Herman, A. Ben-David, and K. J. Thome

A method for numerically solving the equation of radiative transfer in a spherical shell atmosphere is presented. The method uses a conical boundary and a Gauss-Seidel iteration scheme to solve for all orders of scattering along a single radial line in the atmosphere. Tests of the model indicate an accuracy better than 1% for most Earth-atmosphere situations. Results from this model are compared with flat-atmosphere model results for a scattering-only atmosphere. These comparisons indicate that excluding spherical effects for solar zenith angles greater than $85^\circ$ leads to errors larger than 5% at optical depths as small as 0.10.

Key words: Spherical shell radiative transfer, atmospheric radiative transfer.

Introduction

Several methods exist for finding the distribution of scattered radiation in a plane-parallel atmosphere illuminated by a plane-parallel solar beam. In these techniques the atmosphere is assumed to consist of homogeneous plane-parallel layers of infinite extent. This flat-atmosphere approximation is normally adequate for treating radiation traveling in the Earth's atmosphere, and because of computational efficiency, plane-parallel models are used whenever possible. There exist situations for which the plane-parallel model is not applicable. For example, a plane-parallel model cannot address subhorizon solar beams. Significant differences between results from plane-parallel models and spherical models also arise for solar zenith angles greater than $\sim 75^\circ$ and for observation angles within $\sim 10^\circ$ of the horizon, particularly for reflected radiation exiting the top of the atmosphere. Analyses of these situations require a radiative transfer model that considers multiple scattering in a spherical shell atmosphere with altitude-dependent scattering and absorption. A comprehensive model would also include the effects of refraction and polarization.

The differential form of the radiative transfer equation in a spherical shell atmosphere, though simple in form, has no analytical solution. However, numerical solution techniques have since been developed. These include such methods as invariant imbedding, stream approximations and moments of intensity, and Monte Carlo techniques. Whitney et al. and Lenoble review these and other methods in further detail. Approximations that simplify the problem are also used, such as approximate expressions for the mean intensity in a Rayleigh atmosphere; iterative methods based on the Eddington approximation; solving the single scatter exactly and approximating higher-order scattering in terms of this exact single scatter; and a quasi-spherical iterative method utilizing the plane-parallel equations but with all angles and distances corrected for spherical geometry.

The current study addresses cases for incident solar zenith angles between $0^\circ$ and $90^\circ$. Future research, in preparation, will address the case of a subhorizon Sun. The atmosphere is divided into homogeneous spherical shells. The intensity at any point is then a function of five independent variables: three for location in the atmosphere and two for the direction of the radiation. The model includes aerosol scattering as well as gaseous and aerosol absorption, but it neglects polarization and atmospheric refraction. Additionally, an accurate model for the aerosol forward peak has yet to be included. For each shell, the model uses the integrated form of the general radiative transfer equation and computes all distances and angles by assuming spherical geometry.
The technique utilizes a Gauss–Seidel iteration scheme to establish a steady-state solution that includes all significant orders of scattering. Intensities are found at a discrete set of directions at discrete height levels along a selected radial line in the atmosphere. A conical boundary is introduced about this radial line, and approximate solutions are found for each level along selected radii around the boundary. These solutions are used as boundary values in an interpolation scheme to find the required boundary. These solutions are used as boundary values in an interpolation scheme to find the required intensities. This approach permits more versatility and accuracy than previous spherical models that strived for computational efficiency. It is also more computationally efficient than other methods striving for the same accuracy as the current one.

Theory

Figure 1 illustrates the coordinate system used in the current study. The origin of this system is taken to be the center of the planet. Intensities are found at selected heights along the z axis; the z axis is the radius along which the solution is desired. This line is also referred to as the zenith. In other words, the zenith is any radial line from the center of the planet to the top of the atmosphere along which intensities are found. Height above the planet's surface is measured along the z axis and is designated by the variable z. Values for z range from zero at the surface to Zo at the top of the atmosphere. The radius of the planet is designated by Ro. An arbitrary point s in the atmosphere is further defined by the variables ψ and η. The variable ψ is the polar angle between the radial line through s and the z axis, whereas η is the azimuth angle between s and the zero-azimuth plane, defined by the z axis and the incoming solar beam. It should be noted that ψ and η do not represent latitude and longitude but are measured with reference to the principal plane and the radial line along which the intensities are desired. The polar angle θ, measured relative to the z axis, and the azimuth angle φ, measured relative to the zero-azimuth plane, define the line of sight direction. Then the intensity in an arbitrary direction θ, φ and through a point z, ψ, η is represented by \( I(z, \psi, \eta, \theta, \phi) \). Through the use of the previously defined notation for s, this becomes \( I(s, \theta, \phi) \). The solar zenith angle \( \theta_0 \) and solar azimuth angle \( \phi_0 \), which is taken to be zero, define the direction of the incident solar beam.

Following the notation of Chandrasekhar, we may write the integral form of the general radiative transfer equation as

\[
I(s, \theta, \phi) = I(s = 0, \theta, \phi) \exp[-\tau(s, 0)]
+ \int_0^s J(s', \theta, \phi) \exp[-\tau(s, s')] \kappa \rho ds',
\]

(1)

where \( \tau(s, s') \) is the optical depth along the path between the points s and s',

\[
\tau(s, s') = \int_s^{s'} \kappa \rho ds',
\]

(2)

\( I(s, \theta, \phi) \) and \( I(s = 0, \theta, \phi) \) are the total intensities at the points s and s = 0 in the direction \( \theta \) and \( \phi \), s = 0 is the point where the path begins, which can occur at the top or bottom of the atmosphere or at some point within the atmosphere, \( \kappa \) is the mass extinction coefficient with the units of area per unit mass, and \( \rho \) is the density of the attenuating medium. Here \( J(s', \theta, \phi) \) is the source function and for a scattering atmosphere is

\[
J(s', \theta, \phi) = P(s', \theta, \phi, \theta_0, \phi_0) F(s', \theta_0, \phi_0)
+ \int_0^{2\pi} \int_0^{\pi} P(s', \theta, \phi, \theta', \phi') I(s', \theta', \phi')
\times \sin \theta' \, d\theta' \, d\phi',
\]

(3)

where

\[
\int_0^{2\pi} \int_0^{\pi} P(s', \theta, \phi, \theta', \phi') \sin \theta' \, d\theta' \, d\phi' = \omega_0.
\]

(4)

\( P(s', \theta, \phi, \theta', \phi') \) is the phase function describing the angular distribution of scattered light, \( \omega_0 \) is the single-scatter albedo that is unity for conservative scattering, and \( F(s', \theta_0, \phi_0) \) is the reduced solar irradiance at the point s'.

In this study the atmosphere is broken into homogeneous shells. The total intensity at point s is taken to be the sum of single- and multiple-scattering contributions. The term single scattering refers to the first scatter out of the incoming solar beam; multiple scattering refers to the subsequent scattering of the diffuse radiation and does not include single scattering. The sum of the single- and multiple-scattering terms constitutes the total intensity. Thus the total intensity at point s along the level i in the direction \( \theta \) and \( \phi \) is written as

\[
I(s_i, \theta, \phi) = I_{ss}(s_i, \theta, \phi) + I_{ms}(s_i, \theta, \phi).
\]

(5)

The first term on the right-hand side, \( I_{ss}(s_i, \theta, \phi) \), is
the single-scatter intensity, given by

\[ I_{ss}(s_i, \theta, \phi) = \int_0^{s_i} P(s', \theta, \phi; \theta_0, \phi_0) F_s(s', \theta_0, \phi_0) \times \exp[-\tau(s_i, s')] \kappa \rho ds'. \]  

(6)

Here \( I_{ss} \) is also known as the primary scatter and is simply the integral over optical depth of the single scatter at all points \( s' \) transmitted to the point \( s_i \). In the case where the path strikes the surface of the planet, a reflection term is added. Equation (6) assumes that the incident single-scatter intensity at the top of the atmosphere is zero.

The multiple-scatter contribution, \( I_{ms}(s_i, \theta, \phi) \), is similarly the integral over optical depth of the second term in Eq. (3) transmitted to \( s_i \). In the current research the multiple scatter is calculated within each layer, so that at the level \( i \) it is the intensity caused by the scattering of diffuse radiation in the layer between the levels \( i \) and \( (i - 1) \) and an attenuated incident term from the previously calculated multiple scatter at the level \( (i - 1) \). The level \( (i - 1) \) refers to either the level above or below the level \( i \), depending on whether the radiation is traveling downward or upward. With this in mind we write the multiple-scatter contribution at point \( s_i \) as

\[ I_{ms}(s_i, \theta, \phi) = I_{ms}(s_{i-1}, \theta, \phi) \exp[-\tau(s_i, s_{i-1})] \]

\[ + \int_{s_{i-1}}^{s_i} \int_0^{2\pi} \int_0^{\pi} P(s_i, \theta, \phi; \theta', \phi') I(s_i, \theta', \phi') \times \exp[-\tau(s_i, s')] \kappa \rho \sin \theta' d\theta' d\phi' ds', \]

(7)

where the first term on the right-hand side is the attenuated multiple-scatter intensity from the point \( s_{i-1} \) in the direction \( \theta \) and \( \phi \), and the second term is the intensity caused by the scattering of diffuse radiation within the layer into the direction \( \theta \) and \( \phi \). The intensity used in the integral of the second term is the total intensity, and the intensity used for the incident term is the multiple scatter only.

**Method**

The single-scatter term, though complicated by the spherical geometry, is calculated in a straightforward manner by approximating the path integral of Eq. (6) with a layer-by-layer summation. The difficulty in computing the total intensity in a spherical atmosphere is in finding the multiple-scatter contribution, partly because of the lack of horizontal homogeneity. This lack of horizontal homogeneity is caused by the variation of the solar zenith angle at the top of the atmosphere caused by the spherical nature of the problem, as shown in Fig. 2 (i.e., \( \theta_0 \neq \theta_0' \neq \theta_0'' \)). Sphericity also causes parallel paths beginning at the same height to have differing optical paths, as shown in Fig. 2(a). Both of these factors cause the intensity fields at the same height but at varying locations to differ. In a plane-parallel atmosphere, shown in Fig. 2(b), these problems do not occur and the solution need only be found along one vertical axis to describe the intensity field in the entire atmosphere. In a spherical atmosphere, computational problems arise because solutions have to be found over the entire sunlit hemisphere to solve the problem completely.

The current research avoids solving throughout the entire hemisphere by using an approximate method that employs a conical boundary surrounding the \( z \) axis. The size of this conical boundary is defined by the angle \( \psi_0 \), where \( \psi \) is the angle formed by an arbitrary radial line and the \( z \) axis [Fig. 1(a)]. Approximate solutions are found along this boundary, and these solutions are used in an interpolation scheme to solve for the intensity field along the zenith. This procedure reduces the number of computations significantly. The accuracy of the method highly depends on the accuracy of the boundary solution and how errors on the boundary affect the \( z \)-axis solution.

The atmosphere is divided into shells, and the method uses discrete angles. A grid system is set up with the points of the grid defined by the intersections of the zenith and boundary radial lines with the
discrete height levels. The boundary radial lines are at \( \eta \) intervals of 30° about the zenith. Figure 3 shows grid points for the zenith and two of the conical boundary radial lines, corresponding to \( \eta = 0° \) and \( \eta = 180° \). The process of solution begins by computing the single scatter at all grid points on the zenith and boundary for all discrete directions defined by the set of \( \theta \)'s and \( \phi \)'s.

The multiple scatter on the zenith is calculated by iterating to a stable solution in a fashion similar to that of the flat-atmosphere situation.26 Because of the lack of horizontal homogeneity, the intensities at a given level cannot be assumed to be constant around a shell at a given height (points a, b, and c in Fig. 3). Instead, interpolation techniques are used to determine the needed intensities for finding the zenith solution. To compute \( I_{ms}(s_{i-1}, \theta, \phi) \) in Eq. (7), we interpolate between intensities in a given \( \theta \) and \( \phi \) direction at surrounding grid points. Using Fig. 3 as an example, we obtain the intensity at the point \( s_{i-1} \) by means of a quadratic interpolation of the intensities at points a, b, and c. The source integral, the second term on the right-hand side of Eq. (7), is computed by assuming that the total intensity within the layer (i.e., single- plus multiple-scattered light) varies linearly along the path with the value at the point \( s_{i-1} \) obtained by interpolation of grid-point values from the previous iteration. The mass extinction coefficient, density of the attenuating medium, and phase function for a given direction are taken to be homogenous along a shell outside the conical boundary is better than assuming the intensity field to be homogeneous, because the intensity field varies much more rapidly along a shell than the ratio of the multiple to single scatter. This method also permits the known single scatter to be used for added information.

To compute the source term integral for the path within the layer, from the \( (i - 1) \) point outside the cone to the point on the cone boundary, the second term on the right-hand side of Eq. (7), we find it necessary to know intensity as a function of angle along the path of the integral [line \( s_{i-1}(\psi = \psi_0) \) to \( s_i(\psi = \psi_0) \) in Fig. 3]. However, these intensities are not known outside the cone. We therefore approximate the source integral in the following manner. On the zenith (point \( s_i \) in Fig. 3), the source integral is given by

\[
\int_{s_{i-1}}^{s_i} J(s', \theta, \phi) \exp[-\sigma(s_i, s') \kappa \rho ds']
\]

This is simply a restatement of the second term on the right-hand side of the Eq. (7) integral and is solved by using the method previously described. On the cone [point \( s_i(\psi = \psi_0) \) in Fig. 3], we assume the source integral to be the source integral for the zenith modified by the ratios of the path integrations and the
Checking the Model

The model was first tested for errors in the code itself. This was done by ensuring energy conservation in a scattering-only atmosphere ($\omega_0 = 1.0$) by computing the flux divergence assuming each layer in the atmosphere to be a cylinder. Also, model runs performed by using a radius of 638,000 km (i.e., 100 × the radius of the planet used in this study) were compared with flat-atmosphere results for the same atmosphere. Last, intensities were calculated for the case of a small solar zenith angle ($\theta_0 = 5^\circ$) and low optical depth (0.05). Results were compared with flat-atmosphere model results, and differences were examined for consistency with geometrical differences. The results from these three tests indicate the model is free from errors in the code.

The effect of the size of $\psi_0$ for a planet with a radius of 6380 km was examined by comparing results for values of $\psi_0$ ranging from 0.25$^\circ$ to 2.0$^\circ$. For 0.75$^\circ < \psi_0 < 1.25^\circ$, solutions differed from one another by less than 1%. For $\psi_0 < 0.75^\circ$ or $\psi_0 > 1.25^\circ$, solutions diverged rapidly. Thus a value of 1.0$^\circ$ appears appropriate and is used in all calculations presented in this paper.

To test the effect of boundary errors, we positively biased the approximate boundary solutions for $\psi_0 = 1^\circ$ by 10%. The final zenith solution using these biased boundary solutions differed by less than 2.5% from the unbiased case. For nontangent lines of sight (paths that intersect the surface), these differences were less than 1%. An additional check of the boundary approximations was performed by examining the approximate accuracy of the boundary solutions.

Through the proper adjustment of the geometry, a boundary solution to one problem can become a zenith solution for a different problem. These tests indicate that the boundary solution for transmitted light is accurate to better than 3% at all angles, and for $\theta < 80^\circ$ (where $\theta = 0^\circ$ indicates radiation traveling vertically downward, i.e., an observer looking upward) the boundary is accurate to better than 1%. Boundary solutions for reflected light for nontangent lines of sight were accurate to better than 3%.

For tangent lines of sight (an observer looking down from the top of the atmosphere), the differences between the two solutions (i.e., the boundary and the zenith solutions) ranged 3–12%, with the largest differences occurring for $93^\circ < \theta < 95^\circ$ (where a line-of-sight angle of 180$^\circ$ indicates radiation traveling vertically upward along the zenith). These angles are the worst because the paths between the levels $i$ and $(i - 1)$ are the longest at these lines of sight.

Because the errors on the boundary in the computer code used by the model are not biased positively or negatively, and because these errors are much less than the 10% used in testing the boundary’s effects, we conclude that the boundary approximations impose less than a 1% error of the final zenith solution.

To test the model’s accuracy we compared it with
three previously developed models, a quasi-spherical approach\textsuperscript{24} and two Monte Carlo approaches.\textsuperscript{17,28} The transmitted intensity results were compared with those of Asous\textsuperscript{24} and Marchuk et al.\textsuperscript{28} For the reflected intensities, the models of Adams and Kattawar\textsuperscript{17} and of Asous\textsuperscript{24} were used. These comparisons were well within the Monte Carlo technique uncertainties of 3%. Thus the current model is taken to be accurate to at least 3%. Because of the close agreement found between the current research and other models, and because the current approach does not suffer from the statistical fluctuations of the Monte Carlo techniques, the current approach is believed to be accurate to better than 1% for most Earth–atmosphere situations.

Results

Here we present results of comparisons between intensities calculated from the current spherical atmosphere model and those of a flat-atmosphere model using a Gauss–Seidel approach.\textsuperscript{26} Two cases are examined: in one the solar zenith angle is varied for constant optical depth, and in the other the optical depth is varied at constant solar zenith angle. The results for the transmitted light and the reflected light are treated separately.

All of the results presented here assume the top of the atmosphere \( Z_0 \) to be 50 km, a planetary radius \( R_0 \) of 6380 km, and a surface reflectance of zero. The original \( \theta \) angles of 0, 5, 15, \ldots, 165, and 175\textdegree and \( \phi \)

![Fig. 4. Ratio of spherical to flat-atmosphere transmitted intensities at the surface for a scattering atmosphere with an 0.50 optical depth: (a) \( \phi = 0\textdegree \), (b) \( \phi = 0\textdegree \), (c) \( \phi = 180\textdegree \), (d) \( \phi = 180\textdegree \).](image)
angles of 0, 30, 60, ..., 150, and 180° from Herman et al. are used, plus an additional 14 θ angles between 80 and 100° that are varied to delineate maximums and minimums in the intensity field. Twelve η angles are used at 30° intervals, and as mentioned previously, ψ₀ = 1°.

The vertical distribution of scatterers is assumed to follow the 1976 U.S. Standard Atmosphere for a midlatitude winter and 23-km surface visibility. The aerosol relative size distribution is constant with height and follows a Junge power law [dn/dlog(r) = cr⁻ν] with ν = 3.0 and minimum and maximum radii of 0.01 and 5.0 μm, respectively. Aerosols are assumed to scatter as Mie particles with refractive index 1.54 + 0i.

Transmitted Light at the Surface
For an infinitely thick atmosphere or for an infinitesimally thin atmosphere, the downward-scattered intensities at the surface are zero. Thus, initially the scattered light at the surface increases with increasing optical depth. The scattered light will reach a maximum value as a result of this gain mechanism and then decrease as a result of two loss mechanisms: (a) energy from scattering within a layer is attenuated by subsequent layers, and (b) the incident solar energy, which is the source for all scattering events, is reduced as it penetrates into deeper layers and thus less energy is available for scattering. The increase in intensity as a result of increased optical depth is from the integration over optical depth of the source term in the second terms of Eqs. (1) and (7). The decrease of the transmitted intensity with the increase of optical depth is from the reduced transmission in the exp[−τ(s, s')] term. It should be kept in mind that large optical depths can result from both large zenith angles (long geometric paths) and an optically thick atmosphere.

The ratios of the transmitted intensity of the current spherical atmosphere model to a flat-atmosphere model are given in Figs. 4 and 5. Excluding the case of the subhorizon Sun, optical paths are always longer for transmitted light in a flat atmosphere. For small optical depths this leads to larger intensities in a flat atmosphere relative to a spherical atmosphere for the reasons given above. At larger optical depths the loss mechanisms decrease the transmitted intensity in the flat atmosphere faster than in the spherical atmosphere. Thus intensities for the flat atmosphere are smaller at larger path lengths or optical depths caused by reduced transmission. The shorter solar path in the spherical atmosphere also permits deeper penetration of solar energy, and the transmitted intensities for large solar paths will be greater for the spherical atmosphere than for the flat one.

The first case, that of varying Sun angle at constant optical depth, is shown in Fig. 4. Here the vertical optical depth is 0.50 (Rayleigh optical depth of 0.33 and aerosol optical depth of 0.17). The solar zenith angle θ₀ takes values of 5, 35, 65, 75, 80, 85, and 88°, where θ₀ = 0° indicates the Sun is directly overhead on the zenith. Figures 4(a) and 4(b) show results for 5° ≤ θ ≤ 88.9° and ψ₀ = 0°, whereas ψ = 180° is shown in Figs. 4(c) and 4(d). In these figures a line-of-sight angle of 0° indicates that the radiation is traveling vertically downward along the zenith. An azimuth angle of zero° indicates that the radiation is traveling in the same azimuthal direction as the solar beam.

For θ ≤ 75°, Figs. 4(a) and 4(c), the difference between flat and spherical models is less than 4%. It is also noticed that for θ₀ ≤ 35°, the flat-atmosphere results exceed the spherical ones for almost all lines of sight. This is due to the longer
paths of the flat atmosphere enhancing the gain mechanism relative to the spherical one, as discussed previously.

The minimum in the curves at ∼80° is due to the trade-off between the loss and gain mechanisms. For θ < 80°, scattering into the line of sight is more important. Thus because the line-of-sight optical depths are increasing as θ increases, and because this increase occurs faster in the flat atmosphere, the ratio has a decreasing trend. For θ > 80°, the loss mechanism becomes dominant and is larger in the flat atmosphere because of the longer paths. As a result the ratio increases. These factors become more pronounced for φ = 180° because the solar path is larger in this case than it is for φ = 0°. This minimizes the differences caused by solar path bet-

between the flat and the spherical atmospheres (discussed further below), and effects caused by line-of-
sight path differences become more pronounced [Fig. 4(c)].

In Figs. 4(b) and 4(d), the peaks in the curves can be explained by using the above loss and gain mechanisms. For large enough line-of-sight paths, the loss mechanism dominates the gain mechanism. This occurs first in a flat atmosphere and the ratio then increases with the line-of-sight angle. Eventually, the loss mechanisms dominate in the spherical case, and from this point on the ratio decreases with the line-of-sight angle. This effect is more pronounced in the case of larger solar zenith angles, in which the solar attenuation is larger and effects caused by losses are more evident.

Fig. 6. Ratio of spherical to flat-atmosphere reflected intensities at the top of the atmosphere for a scattering atmosphere with an 0.50 optical depth: (a) φ = 0°, (b) φ = 0°, (c) φ = 180°, (d) φ = 180°.
For long solar optical paths, the effects of sphericity become less noticeable as a result of the large amount of attenuation of the incoming energy. This explains why the ratios in Figs. 4(c) and 4(d) are closer to 1.0 than for the corresponding curves in Figs. 4(a) and 4(b). However, there is always some difference between the flat and spherical atmosphere intensities because of the shorter solar path in the spherical atmosphere, and thus the spherical to flat ratio will always exceed unity for large enough vertical optical depths.

The case for varying optical depth with constant Sun angle is shown in Fig. 5. A pure Rayleigh atmosphere is used with optical depths of 0.1, 0.2, 0.4, 0.6, and 0.8. The solar zenith angle \( \theta_0 \) is 85°. Figure 5(a) shows the ratio for \( \phi = 0^\circ \), and the ratio for \( \phi = 180^\circ \) is given in Fig. 5(b). The shapes of these curves can be explained by using the previous arguments and will not be repeated. The most important effect is for \( \theta \) less than \( \sim 75^\circ \), where the ratio increases as the optical depth increases. This is a result of larger loss mechanisms for the flat-atmosphere case. As optical depth increases to larger and larger values, the differences between the flat and spherical atmospheres become minimized. Thus the increase in the ratio is larger at smaller optical depths. For near-horizon lines of sight, the interplay of the loss and gain mechanisms complicates this discussion.

Reflected Light At the Top of the Atmosphere

In a conservative atmosphere (i.e., no absorption), the total reflected energy increases with increasing optical depth. As the optical depth increases, the reflected energy increases at the expense of the transmitted energy. This qualitative explanation for the reflected energy is useful when one examines the reflected intensities as a function of geometry (i.e., solar angle and line-of-sight angle). The same two cases used above for the transmitted case are used to examine the reflected light exiting the top of the atmosphere and are presented in Figs. 6 and 7. Here a line-of-sight angle \( \theta \) of 180° indicates radiation traveling vertically upward along the zenith. The inadequacies of a flat-atmosphere model caused by the lack of tangent lines of sight (paths that do not intersect the planet's surface) become readily apparent when one examines the results of these comparisons. Because of tangency, the longest path in a spherical atmosphere occurs for that line of sight that is tangent to the surface (\( \theta = 97.15^\circ \) with \( Z_0 = 50 \) km). In the flat atmosphere, the longest path occurs in the limit as \( \theta \) approaches 90°. For \( \theta \) less than \( \sim 95^\circ \) (for \( Z_0 = 50 \) km), the flat-atmosphere path lengths are longer than the spherical ones, whereas the spherical path lengths are longer for all other line-of-sight angles. Finally, as in the transmitted case, solar paths are always shorter in the spherical atmosphere.

The case of constant optical depth (aerosol optical depth of 0.17 and Rayleigh optical depth of 0.33) is shown in Fig. 6. The zero azimuth case is shown in Figs. 6(a) and 6(b), whereas the case for \( \phi = 180^\circ \) is shown in Figs. 6(c) and 6(d). The dominating features in Fig. 6 are the sharp peaks caused by tangent effects. As described above, these effects result in path lengths in the spherical atmosphere that are much smaller than those in the flat atmosphere for angles less than \( \sim 95^\circ \). Thus, using the same arguments presented for the transmitted light case, we see that the intensities in the spherical atmosphere case are smaller, and the ratio is less than one. These effects become greater as the solar zenith angle increases, except in Fig. 6(d). In this case for an azimuth angle of 180° (i.e., looking away from the Sun) and for the Sun near the horizon, some of the
lines of sight cross the planet’s shadow, and there is no single-scatter contribution [Eq. (6)] from this portion of the path in the spherical atmosphere model. Because there is no shadow in a flat-atmosphere model, the single-scatter contribution will always exceed the single-scatter component of the spherical atmosphere model, and the ratio will be smaller. The increase in the peaks as \( \theta_0 \) increases arises from less solar energy being available to be scattered in the flat atmosphere (as discussed above), and the longer paths in the spherical case permitting more light to be scattered into the line of sight. The curves are smoother for \( \phi = 180^\circ \) because of the minimization of spherical effects in the solar path (as previously discussed) in addition to shadow effects.

Figure 7 shows the varying optical depth case. Results for \( \phi = 0 \) are given in Fig. 7(a), whereas Fig. 7(b) shows the case for \( \phi = 180^\circ \). The most notable features are (a) the peaks in the ratio decrease and move to lower line-of-sight angles with increasing optical depth; (b) for \( \theta \) greater than \( \sim 110^\circ \), the ratio increases with increasing optical depth; and (c) the ratio increases rapidly for small changes in vertical optical depth for tangent lines of sight. As the optical depth increases the peak intensity occurs at smaller line-of-sight angles because of the attenuation along the line of sight. Because the intensity peak has shifted to lower angles, the ratio peak shifts as well. The increase in the ratio with increasing optical depth at larger line-of-sight angles is due to an increase of scattering into the line of sight. Because the paths are longer in the spherical atmosphere, intensities should increase faster relative to those of a flat atmosphere. The cause for the smoothness of the curves in Fig. 7(b) has been previously explained.

Conclusions

A method for numerically solving the equation of radiative transfer in a spherically symmetric atmosphere is presented. The method uses a conical boundary with an interpolation scheme to solve for the intensity field along a single radial line in the atmosphere. Tests of the model indicate that it is accurate to better than 1% for most Earth-atmosphere situations.

Comparisons with flat-atmosphere results are presented for two cases. The first case examines the differences in a scattering-only atmosphere with a total optical depth of 0.50 and by using various solar zenith angles. The second case examines the effect of changing optical depth at a solar zenith angle of 85°. Differences greater than 10% are found in the transmitted intensities at the surface for all lines of sight when the solar zenith angle is greater than 85° and spherical effects are neglected. For an optical depth of 0.50, a solar zenith angle greater than 80°, and view angles within 10° of the horizon, spherical effects must be included to avoid errors of the order of 5%. For a solar zenith of 85°, errors of 5% are found for optical depths as small as 0.10.

These flat-atmosphere comparisons also indicate the need for including spherical effects when one examines the intensity field at the top of the atmosphere. For lines of sight that do not strike the Earth’s surface, there is no true comparison with the flat-atmosphere model. Thus for a 50-km-thick atmosphere and 6380-km-radius planet, a spherical model must be used for lines of sight greater than 82° from nadir. There is a similar interplay between solar zenith and view angle as described above for the reflected intensities as well. Again, for solar zenith angles greater than 85°, spherical effects must be included to obtain results accurate to better than 10% at all view angles.

These results are obtained by using a surface reflectance of zero. Future research will include studies to examine the ratio of spherical to flat atmosphere model results as a function of changing surface reflectance. Also planned are more in-depth studies of the type presented here. Future research will include examining methods for the subhorizon Sun. This will eventually permit further research investigating the light scatter at twilight.

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References