A model for subglacial flooding through a preexisting hydrological network during the rapid drainage of supraglacial lakes

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Abstract

Increasingly large numbers of supraglacial lakes form and drain every summer on the Greenland Ice Sheet. Some of these lakes drain rapidly within the timescale of a few hours, and the vertical discharge of water during these events may find a preexisting film of water potentially within a distributed drainage system of linked cavities. Here we present a model for subglacial flooding applied specifically to such circumstances. One of many interesting results we find is that water flows in from the far field prior to the arrival of flooding. We systematically evaluate the effect of initial ice/bed opening on the degree of perturbation to the subglacial system. Of particular importance, we find that floods propagate much faster and vertical displacements are much greater for larger openings. For 10 and 1 cm of initial opening, for example, floods travel about 68% and 50% farther than in the fully coupled ice/bed scenario after 2 h of drainage, respectively. For the same choices of initial opening, the elastostatic displacement at the injection point is about 1.39 and 1.26 times that of the fully coupled scenario. Using the framework with a preexisting water film results in avoiding the pressure singularity that is inherent to classical hydrofracture models, thus opening an avenue for integrating the likes of our model within continuum subglacial hydrological models. Furthermore, we foresee that the theory presented can be used to potentially infer subglacial hydrological conditions from surface observables.

1. Introduction

Observed diurnal [Shepherd et al., 2009] and seasonal accelerations [e.g., Bartholomew et al., 2010] of land-terminating sectors of the Greenland Ice Sheet (GrIS) are believed to be controlled by routing of surface meltwater to the bed [e.g., Zwally et al., 2002; Das et al., 2008; Palmer et al., 2011]. However, it is not simply the mean meltwater supply but its variability [e.g., Bartholomaus et al., 2008; Schoof, 2010] that is responsible for the faster ice flow. A sudden increase in otherwise steady melt supply, for example, is accommodated by the subglacial hydrological system by spontaneously reducing the local effective pressure (or, in other words, by raising the subglacial water pressure) before the system can adjust its geometry toward efficient drainage. Such a reduction in effective pressure during and after the additional melt supply tends to decouple the ice and its bed, thus promoting faster basal sliding.

In this context, the formation of supraglacial lakes (SGLs) and the nature of their drainage are of paramount importance as they seem to control the melt supply variability to subglacial hydrological systems. Increasingly large numbers of SGLs form on the GrIS, whose ablation regions migrate every summer toward the ice sheet interior [e.g., Sundal et al., 2009; Selmes et al., 2011; Liang et al., 2012]. This trend will likely be amplified in the future [Leeson et al., 2014], as increased surface melting [e.g., Parizek and Alley, 2004] and more frequent rain events [Schuenemann and Cassano, 2010] are expected in an ongoing warming scenario. Presently, more than 13% of SGLs drain rapidly [Selmes et al., 2011], with typical timescale of a few hours, by fracturing through kilometer-thick ice [e.g., Das et al., 2008; Doyle et al., 2013]. The remaining lakes may percolate through to the bed relatively slowly (typical timescale of a few days) or refreeze in situ or en route to the bed in the following winter. We are interested in modeling subglacial flooding during the former, short-timescale drainage events.

While slow drainage of SGLs through nearby preexisting moulins may sufficiently lower effective pressure to yield considerable basal sliding, rapid drainage events simply overwhelm the subglacial system, thus leading to much larger vertical uplifts and ice sheet accelerations [e.g., Zwally et al., 2002; Shepherd et al.,...]
2009; Tedesco et al., 2013]. Large horizontal (kilometer-scale) fractures with a few active moulins (diameters of ≈10 m) and scattered ice blocks (diameters of several meters) are typical surface features observed in situ after drainage events as reported, for example, by Das et al. [2008], Doyle et al. [2013], and Tedesco et al. [2013]. Such perturbations in ice sheet geometry will likely be amplified in the future, as we expect an increase in the frequency of drainage events due to climate warming. Furthermore, the potential positive feedback between melt supply variability, strongly controlled by the formation and drainage of SGLs, and dynamic thinning of the ice sheet may also be expected to strengthen in the future. In order to improve our predictions of GrIS dynamics through AD 2100 and beyond, it is therefore crucial to appropriately account for SGL drainage events and associated subglacial floodings in a coupled system of ice flow and subglacial hydrological models.

The contemporary state-of-the-art models of subglacial hydrology [e.g., Schoof et al., 2011; Schoof et al., 2012; Hewitt et al., 2012; Werder et al., 2013] generally describe the drainage mechanism in a normal pressure regime, i.e., with water pressure \( p \in [0, \sigma_0] \) where \( \sigma_0 \) is the ice overburden pressure. During the rapid vertical drainage of SGLs, however, water pressure exceeds the overburden pressure (i.e., effective pressure becomes negative), assuming a continuous column of water in the mounlin [e.g., Tsai and Rice, 2010, 2012]. Consequently, the base of the ice sheet uplifts elastostatically in order to accommodate the room for added water. For brevity, we term the water pressure in excess of overburden pressure, i.e., \( p - \sigma_0 \), as the “excess pressure” and denote it by \( \Delta p \).

Tsai and Rice [2010, 2012] employ an LEFM approach to evaluate how the fully coupled ice/bed interface is decoupled by turbulent hydraulic fracture during the rapid drainage of SGLs (see Figure 1a). One of the important features of their solutions is that pressure singularities exist at the crack tips, although these are relatively weaker than the classical elastic square root singularity [Desroches et al., 1994]. As a result, the hydrofracture models of Tsai and Rice [2010, 2012] are not compatible with contemporary hydrological models. It might also be difficult to parameterize the hydrofracture solution in the latter models, particularly for \( \Delta p \) near the crack tips.

In order to avoid the solution singularities, we hypothesize the presence of a thin film of water underneath the prospective drainage site (see Figure 1b). Our model framework (i.e., with a preexisting subglacial water film) is consistent with the spirit of contemporary continuum hydrological models [e.g., Flowers and Clarke, 2002; Creyts and Schoof, 2009; Le Brocq et al., 2009; Schoof, 2010; Hewitt, 2011; Pimentel and Flowers, 2011; Schoof et al., 2012; Hewitt et al., 2012; Werder et al., 2013; de Fleuriéon et al., 2014]. In reality, the hypothesized film of water may be interpreted as a preexisting subglacial hydrological network (see, e.g., Fountain and Walker [1998] for a review), particularly one that is distributed [e.g., Kamb, 1987] rather than channelized [e.g., Röthlisberger, 1972]. It may be a thin sheet of water [e.g., Weertman, 1972] as observed, for example, by Johannesson [2002] and Magnússon et al. [2007], a system of linked cavities [e.g., Walker, 1986; Kamb, 1987] or a shallow subglacial lake [e.g., Palmer et al., 2013]. Theoretically, we may also interpret the initial water film as a microscopic interfacial layer of premelted water [e.g., Dash et al., 2006] or as a film of water produced locally by the regelation-sliding mechanism [e.g., Nye, 1973]. The consideration of a thin film of water underneath the prospective drainage site therefore seems to characterize the subglacial conditions of warm-based interior ablation regions of the GrIS reasonably well. However, our model may be less relevant very near the margins of the ice sheet where efficient channelized drainage systems are expected to be sustained perennially [Chandler et al., 2013].

Here we present a model for subglacial flooding through a preexisting hydrological network during the rapid drainage of SGLs. In section 2, we present the governing equations and relevant conditions/criteria...
Figure 1. Model schematics: (a) the Tsai and Rice [2010] problem and (b) the problem considered in this research. Left shows the initial model setup for each case: the drained water may find a fully coupled ice/bed area (top) or a preexisting subglacial hydrological network (bottom). In the latter model, an infinite extent of the opening is assumed, which has uniform initial opening, i.e., \( w(x,0) = w_i \) and is fully filled with water. Water is assumed to be at rest, i.e., \( q(x,0) = 0 \), and zero excess pressure, i.e., \( \Delta p(x,0) = 0 \). Right illustrates the hypothesized configuration of subglacial flooding some time after the inception of SGL drainage: the subglacial crack is driven open by the hydraulic fracture (top); the preexisting opening evolves due to the elastostatic deformation of ice and bed (bottom). It is reasonable to assume that \( \Delta p \) remains more or less constant at the point of injection during the rapid SGL drainage, i.e., \( \Delta p(0,t) \approx \Delta p_0 = \text{const} \). In the Tsai and Rice [2010] model, the crack tip is defined to have zero total opening, i.e., \( w(|x| = L, t) = 0 \), and thus zero flux, i.e., \( q(|x| = L, t) = 0 \). In order for the crack to propagate, the fracture criterion that the stress intensity factor, \( K_I \), must be equal to the material toughness, \( K_{IC} \), should be satisfied. In our model, the required far field condition, i.e., as \( |x| \to \infty \), is that the field variables must remain unperturbed for all the time. (Note that the “flood front” is defined, for ease of discussion, as the location at which flow reverses. Furthermore, the ice thickness does not change in either model, and surface uplift near the moulin would indeed be expected, although it is not depicted in the schematics.)

required to solve the problem. Both elasticity and fluid flow equations are presented in detail. In section 3, we describe the spectral approach of series minimization employed to solve the governing equations. In section 4, model results are presented and their physical significance is discussed. We show that lowering of water pressure, reversal in flow direction and minor subsidence of the ice surface prior to the arrival of flooding are unique characteristics of our model. The effects of initial ice/bed opening (cf. Figure 1b) on the rate of flood propagation and along-flow profiles of field variables are also analyzed. In section 5, we differentiate our model with respect to the hydrofracture model of Tsai and Rice [2010] that is strictly applied to the fully coupled ice/bed scenario. In section 6, we discuss the scope and limitations of our model, outline future work needed for its improvement, and highlight the implications. Finally in section 7, we summarize key conclusions of this research.

2. Governing Equations

We model subglacial flooding along an idealized plane strain horizontal opening during the rapid drainage of SGLs (cf. Figure 1b). This rapid drainage is a short timescale phenomenon (of order \( \approx 2 \) h), during which elastic response of ice sheet should dominate over its slow viscous adjustment. We therefore model the ice and bed as purely elastic. We hypothesize that the drained SGL water finds a preexisting subglacial hydrological network, such as a thin film of water [e.g., Weertman, 1972; Le Brocq et al., 2009] potentially within a distributed system of linked cavities [e.g., Walder, 1986; Kamb, 1987]. This preexisting network is assumed to be fully filled with water, which is at rest and zero excess pressure everywhere. We further assume that the bed and the base of the ice sheet are horizontal. Flow of subglacial water during the rapid drainage event is turbulent [e.g., Tsai and Rice, 2010] and hence treated accordingly.

Here we present the governing equations and boundary conditions required to model the plane-strain problem of subglacial flooding outlined above.

2.1. The Elasticity Equation

The elasticity equation required to solve our problem can be derived from the complex potential theory of plane elasticity [e.g., Muskhelishvili, 1953; England, 1971]. We consider the ice (i.e., \( y \geq 0 \)) and bed (i.e., \( y \leq 0 \)) as two elastic half planes. (See Figure 1 for the positive sense of coordinate axes \( x \) and \( y \).) For elastostatic
half-plane problems, the externally applied normal stress, $\Delta p(x, t)$, at time $t$ and the resulting vertical displacement, $\Delta w(x, t)$, on $y = 0$ are related as follows:

$$\frac{\partial \Delta w(x, t)}{\partial x} = \frac{2}{\pi E'} \int_{-\infty}^{\infty} \frac{\Delta p(s, t)}{s-x} \, ds,$$

where $E' = E/(1-\nu^2)$ is the effective modulus of elasticity, $E$ is Young's modulus of elasticity, and $\nu$ is Poisson's ratio. Given the (analytic) pressure profile, the gradient in the vertical displacement can be evaluated analytically. To avoid the singularity intrinsic to equation (1), however, we must evaluate the integral in a Cauchy principal value sense.

While a full derivation of equation (1) using a complex potential method would be lengthy (and is, for example, given in Muskhelishvili [1953]), we provide here a brief intuitive explanation: A single mathematical dislocation (step in displacement) is well known to have a stress field that decays as $1/r$, where $r$ is the radial distance away from the dislocation. The pressure along a line with a given dislocation density can then be described as an integral over $1/(s-x)$ multiplied by the dislocation density (which, in the context here, would be $\partial \Delta w/\partial x$). Equation (1) can then be derived by inserting the appropriate moduli and inverting the Hilbert transform expression [e.g., Weertman, 1996].

### 2.2. The Fluid Flow Equations

The turbulent Manning-Strickler relation [Manning, 1891; Strickler, 1981] relates the local gradient in excess pressure, $\partial \Delta p/\partial x$, to the depth-averaged fluid velocity, $v(x, t)$, and total ice/bed opening, $w(x, t)$. (Due to the symmetric nature of the problem, we hereinafter confine our attention to the region $x \in [0, \infty]$ unless stated otherwise.) If fluid flows in the positive $x$ direction, the Manning-Strickler relation is given by [Manning, 1891; Strickler, 1981]

$$-\frac{\partial \Delta p(x, t)}{\partial x} = \frac{1}{4} f_{D} \rho k^{1/3} \frac{v(x, t)^2}{w(x, t)^{4/3}},$$

where $f_{D}$ is the Darcy-Weisbach friction factor coefficient [e.g., Moody, 1944], $\rho$ is the water density, and $k$ is the Nikuradse channel wall roughness [e.g., Rubin and Atkinson, 2001]. Typical values for these parameters/ constants are listed in Table 1.

The total ice/bed opening can be expressed as the sum of the initial opening, $w_{i}$, and the total elastostatic opening, $w_{e}$. As the opening is located along the interface of two dissimilar materials (ice and bed; cf. Figure 1b), elastostatic opening of both half planes should be considered. We express

$$w(x, t) = w_{i} + w_{e}(x, t) = w_{i} + \frac{\xi}{\pi} \Delta w(x, t),$$

### Table 1. Constants and Parameters Used in This Study

<table>
<thead>
<tr>
<th>Constant/Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus of (ice) elasticity</td>
<td>$E$</td>
<td>$6.20 \times 10^{9}$</td>
<td>Pa</td>
<td>at $-5^\circ$C</td>
</tr>
<tr>
<td>Young's modulus of bed material</td>
<td>$E_{bed}$</td>
<td>$59.8 \times 10^{9}$</td>
<td>Pa</td>
<td></td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu$</td>
<td>0.3</td>
<td>–</td>
<td>for both ice and bed</td>
</tr>
<tr>
<td>Darcy-Weisbach friction factor</td>
<td>$f_{D}$</td>
<td>0.143</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>Opening wall roughness</td>
<td>$k$</td>
<td>0.01</td>
<td>m</td>
<td></td>
</tr>
<tr>
<td>Water density</td>
<td>$\rho$</td>
<td>1000</td>
<td>kg m$^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Ice density</td>
<td>$\rho_{ice}$</td>
<td>910</td>
<td>kg m$^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Ice thickness</td>
<td>$H_{ice}$</td>
<td>980</td>
<td>m</td>
<td>$\approx H$ (height of water)</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>$g$</td>
<td>9.81</td>
<td>m s$^{-2}$</td>
<td></td>
</tr>
<tr>
<td>Inlet excess pressure</td>
<td>$\Delta p_{0}$</td>
<td>$0.87 \times 10^{6}$</td>
<td>Pa</td>
<td>$(\rho - \rho_{ice}) g H$</td>
</tr>
<tr>
<td>Characteristic timescale</td>
<td>$t_{0}$</td>
<td>2</td>
<td>h</td>
<td>e.g., Tedesco et al. [2013]</td>
</tr>
<tr>
<td>Characteristic opening</td>
<td>$w_{0}$</td>
<td>0.6097</td>
<td>m</td>
<td>equation (16)</td>
</tr>
<tr>
<td>Characteristic length scale</td>
<td>$s_{0}$</td>
<td>$4.77 \times 10^{3}$</td>
<td>m</td>
<td>equation (12)</td>
</tr>
<tr>
<td>Characteristic flux</td>
<td>$q_{0}$</td>
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<td>m$^{2}$ s$^{-1}$</td>
<td>equation (20)</td>
</tr>
<tr>
<td>Characteristic velocity</td>
<td>$v_{0}$</td>
<td>0.662</td>
<td>m s$^{-1}$</td>
<td>equation (21)</td>
</tr>
</tbody>
</table>

*All values are taken from Tsai and Rice [2010], unless otherwise stated.*
where $\Delta w$ is the elastostatic opening of the more compliant material (i.e., ice), and $\xi \approx 1.04$ is a factor accounting for the bimaterial ice/bed opening (cf. Appendix A).

Next, we define $q(x, t) = \nu(x, t)w(x, t)$, where $q(x, t)$ is the fluid flux. Equation (2) can alternatively be written in the following general form:

$$q(x, t) = -2\chi w(x, t)^{2/3} \left[ \frac{-1}{f_0^3k^{1/3}} \left( \frac{\partial \Delta p(x, t)}{\partial x} \right)^{1/2} \right],$$

(4)

where

$$\chi = \text{sgn} \left( \frac{\partial \Delta p}{\partial x} \right)$$

(5)

is a signum function required to track the sign of the excess pressure gradient. Notice that $\chi$ appears twice in equation (4): once inside brackets to ensure that the flux is always a real number; and a second to determine the direction of flux. Note also that if the excess pressure gradient is negative (i.e., $\chi = -1$), the flux is positive (i.e., fluid flows in the positive $x$ direction). Similarly, if the excess pressure gradient is positive, the flux is negative (i.e., fluid flows in the negative $x$ direction).

Assuming that water is incompressible, the local continuity equation is given by

$$\frac{\partial w(x, t)}{\partial t} + \frac{\partial q(x, t)}{\partial x} \equiv \xi \frac{\partial \Delta w}{\partial t} + \frac{\partial q}{\partial x} = 0.$$  

(6)

Integrating equation (6) from arbitrary $x$ to $\infty$ and imposing $q(x \to \infty, t) = 0$ (cf. section 2.3), we obtain

$$q(x, t) = \frac{\xi}{\xi} \int_x^\infty \Delta w(x, t) \, dx.$$  

(7)

Equations (4) and (7) can then be combined to remove $q(x, t)$ from explicitly appearing as follows:

$$\xi \frac{\partial}{\partial t} \int_x^\infty \Delta w(x, t) \, dx = -2\chi \left[ w^2 + w \Delta w \right]^{5/3} \left[ \frac{1}{f_0^3k^{1/3}} \left( \frac{\partial \Delta p(x, t)}{\partial x} \right)^{1/2} \right].$$  

(8)

Note that both the left-hand side (LHS) and right-hand side (RHS) represent the fluid flux.

### 2.3. Boundary Conditions and Criteria

The elastostatic equation (equation (1)) and fluid flow equation (equation (8)) can be solved for the two unknown field variables $\Delta p(x, t)$ and $\Delta w(x, t)$ if appropriate initial and boundary conditions are provided. Once these are calculated, the remaining unknown variables $q(x, t)$ and $\nu(x, t)$ can be recovered from equations (4) or (7) and the total opening $w(x, t)$ from equation (3).

Equation (1) automatically satisfies the criterion that stresses are bounded as $z \to \infty$, where $z = \sqrt{x^2 + y^2}$ is an arbitrary point on the half plane. For rotational force to vanish as $z \to \infty$, it is sufficient to assume that the external stresses applied on $y = 0$ are symmetric about $x = 0$, i.e., $\Delta p(x, t) = -\Delta p(-x, t)$. For displacement to remain finite as $z \to \infty$, the resultant of the external stresses must be zero, i.e.,

$$\int_0^\infty \Delta p(x, t) \, dx = 0.$$  

(9)

Besides these criteria, the following boundary conditions must be satisfied. At the point of injection, i.e., $x = 0$, excess pressure is assumed to remain constant for all time $\Delta p(x = 0, t) = \Delta p_0$ [e.g., Tsai and Rice, 2010, 2012]. As $x \to \infty$, the field variables remain unperturbed; specifically, $\Delta w(x \to \infty, t) = 0$, $\Delta p(x \to \infty, t) = 0$, and $q(x \to \infty, t) = 0$. Note that the last condition has already been satisfied while deriving equation (7).

Finally, it needs to be mentioned that the fluid flux is discontinuous at the point of injection and due to the problem symmetry given by $q(0^+, t) = Q_0/2$ and $q(0^-, t) = -Q_0/2$, where $Q_0$ is the total fluid injected at time $t$. Equation (4) thus implies that the following inequality must hold at $x = 0$:

$$\frac{\partial \Delta p}{\partial x} \bigg|_{x=0^+} = -\frac{\partial \Delta p}{\partial x} \bigg|_{x=0^-} < 0.$$  

(10)
3. Method

It is possible to employ traditional finite difference or finite element approaches to solve our problem, but numerical evaluation of the singular integral equation (equation (1)) in a Cauchy principal value sense is not straightforward in these cases. Instead, we find it convenient to use a boundary element method; particularly, we employ spectral approach of series minimization as in, for example, Spence and Sharp [1985], Adachi and Detournay [2002], and Tsai and Rice [2010, 2012].

We seek solutions for the nondimensionalized primary field variables by expressing these as series [e.g., Boyd, 2001]. Basis functions for the normalized pressure are defined by combining rational Chebyshev functions appropriately. This allows us to derive the associated basis functions for the normalized displacement by analytically integrating the elasticity equation (equation (1)). The unknown coefficients appearing in the series are finally obtained by satisfying the fluid flow equation (equation (8)).

3.1. Scaling

Before solving the governing equations, we present a scaling analysis that provides an intuition for how the solution is expected to behave. In this simple scaling analysis, we nondimensionalize the primary variables as follows:

\[ \hat{x} = \frac{x}{x_0}, \quad \hat{t} = \frac{t}{t_0}, \quad \Delta \hat{w} = \frac{\Delta w}{w_0}, \quad \Delta \hat{p} = \frac{\Delta p}{\Delta p_0}, \quad (11) \]

where hatted variables are nondimensional and variables with subscript zero are characteristic scales of the respective original variables. Excess pressure is scaled by the known pressure at the point of injection. Time is scaled by the typical timescale over which the event (i.e., SGL drainage) occurs. Characteristic values for the length scale and elastostatic opening are defined below.

Defining

\[ x_0 = \frac{w_0 E'}{\Delta p_0}, \quad (12) \]

Equation (1) can be written in the following nondimensional form:

\[ \frac{\partial \Delta \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}} = \frac{2}{\pi} \int_{-\infty}^{\infty} \hat{p}(\hat{s}, \hat{t}) \hat{s} - \hat{x} \, d\hat{s}. \quad (13) \]

Similarly, equation (8) reduces to the following nondimensional form:

\[ \frac{\xi}{\xi} \int_{0}^{\infty} \Delta \hat{w}(\hat{x}, \hat{t}) \, d\hat{x} = -\frac{2\chi}{t_0^{1/2}} [\hat{w}_i + \xi \Delta \hat{w}(\hat{x}, \hat{t})]^{5/3} \left\{ \frac{1}{\chi} \frac{\partial \Delta \hat{p}(\hat{x}, \hat{t})}{\partial \hat{x}} \right\}^{1/2}, \quad (14) \]

provided that

\[ \frac{w_0 \chi_0}{t_0} = \frac{w_0^{5/3}}{\chi_0} \left\{ \frac{1}{\rho k^{1/3}} \right\}^{1/2} \left( \frac{\Delta \hat{p}_0}{\chi_0} \right)^{1/2}. \quad (15) \]

Note that \( \hat{w}_i = w_i/w_0 \) is the nondimensionalized initial opening. Equations (12) and (15) can be solved for \( w_0 \) to yield the characteristic value of elastostatic opening:

\[ w_0 = \left( \frac{\Delta \hat{p}_0}{\rho k^{1/3}} \right)^{1/5} t_0^{6/5}. \quad (16) \]

For the sake of completeness, we also nondimensionalize equation (9) as

\[ \int_{0}^{\infty} \Delta \hat{p}(\hat{x}, \hat{t}) \, d\hat{x} = 0, \quad (17) \]

and the relevant inequality of equation (10), i.e., only for \( \hat{x} \in [0, \infty) \), as

\[ \frac{\partial \Delta \hat{p}}{\partial \hat{x}} \bigg|_{\hat{x} = 0^+} < 0. \quad (18) \]
The nondimensionalized boundary conditions (that are yet to be satisfied) become (i) \( \hat{\Delta p}(\hat{x} = 0, \hat{t}) = 1; \)
(ii) \( \hat{\Delta p}(\hat{x} \to \infty, \hat{t}) = 0; \) and (iii) \( \hat{\Delta \dot{w}}(\hat{x} \to \infty, \hat{t}) = 0. \)

Finally, other variables are scaled as follows:

\[
\hat{q} = \frac{q}{q_0}, \quad \hat{\psi} = \frac{\psi}{v_0}, \quad \hat{\dot{w}} = \frac{\dot{w}}{w_0}, \quad \hat{\dot{w}}_e = \frac{\dot{w}_e}{w_0}.
\]

Nondimensionalized flux, i.e., \( \hat{q} \), is given by either side of equation (14) and the corresponding characteristic value by either side of equation (15), i.e.,

\[
q_0 = \frac{w_0 x_0}{t_0} = \left[ \frac{\Delta p_0^{17}}{\rho^6 k E t^{13}} \right]^{1/6} t_0^{1/6}.
\]

Similarly, we may write \( \hat{\psi} = \hat{q}/\hat{\dot{w}} \) and

\[
v_0 = \frac{q_0}{w_0} = \left[ \frac{\Delta p_0^7}{\rho^3 k E^2} \right]^{1/5} t_0^{1/5}.
\]

It is also useful to express the relation between openings (equation (3)) in the following nondimensional form:

\[
\hat{w} = \hat{w}_l + \hat{w}_e = \hat{w}_l + \xi \Delta \hat{w}.
\]

In summary, solutions of the governing equations (equations (13) and (14)) will depend on the choice of initial ice/bed opening. Equations (12), (16), (20), and (21) provide an intuition for how the characteristic length, opening, flux and velocity are scaled to the excess pressure, characteristic time, and the material parameters. For example, \( x_0 \) scales as \( \Delta p_0 \) to the power 7/5, as \( t_0 \) to the power 6/5, inversely as \( E \) to the power 4/5, inversely as \( p \) to the power 3/5, and inversely as \( k \) to the power 1/5 (cf. equations (12) and (16)).

### 3.2. Series Expansion and Minimization

We express the solution for \( \hat{\Delta p}(\hat{x}, \hat{t}) \) as a series of the following form, whose basis functions are defined from rational Chebyshev functions on the infinite interval \((-\infty, \infty)\) [e.g., Boyd, 1987, 2001],

\[
\hat{\Delta p}(\hat{x}, \hat{t}) = \frac{1}{D} \sum_{n=0}^{2N} a_n \hat{\Delta \hat{p}}_n(\hat{x}) = \frac{1}{D} \left[ a_0 \hat{\Delta \hat{p}}_0(\hat{x}) + \sum_{n=1}^{N} a_{2n-1} \hat{\Delta \hat{p}}_{2n-1}(\hat{x}) + \sum_{n=1}^{N} a_{2n} \hat{\Delta \hat{p}}_{2n}(\hat{x}) \right].
\]

where \( \hat{\Delta \hat{p}}_n(\hat{x}) \) are basis functions, \( a_n \) are real coefficients, and \( D \) is a function of \( a_n \).

For \( m = 1, 2, 3, \ldots \), we define \( \hat{\Delta \hat{p}}_m(\hat{x}) \) as follows:

\[
\hat{\Delta \hat{p}}_{2m-1}(\hat{x}) = \frac{(2m + 1)G_4(\hat{x})}{4} - \frac{G_{2m+1}(\hat{x})}{2},
\]

\[
\hat{\Delta \hat{p}}_{2m}(\hat{x}) = \frac{G_{4m+1}(\hat{x})}{4(m + 1)} - \frac{G_{2m}(\hat{x})}{2}.
\]

where

\[
G_{2m}(\hat{x}) = TB_{2m}(\hat{x}) - TB_0(\hat{x}),
\]

and \( TB_m(\hat{x}) \) (for \( n = 0, 1, 2, \ldots \)) are the rational Chebyshev functions on the infinite interval \((-\infty, \infty)\) that are symmetric about \( x = 0 \), thus ensuring that \( \hat{\Delta \hat{p}}(\hat{x}, \hat{t}) \) is an even function (cf. section 2.3).

Reasons for choosing rational Chebyshev functions and combining these (according to equations (24)–(26)) to form the basis set of the series in equation (23) are the following: There are only a limited number of alternatives to \( TB_n(\hat{x}) \) for \( \hat{x} \in (-\infty, \infty) \), including the sinc and Hermite functions [e.g., Boyd, 2001]. However, for the slowly decaying functions which appear in our work, these alternatives would require a large number of terms in a series expansion, thus making the computation inefficient. (Some time after the injection of water, \( \Delta \hat{p}(\hat{x}, \hat{t}) \) may perturb the subglacial system over sufficiently large distance; cf. section 4.) Functions \( TB_n(\hat{x}) \) provide the flexibility to scale the basis set by tuning a “mapping parameter” appropriately. \( TB_n(\hat{x}) \)
are related to the Chebyshev polynomials of the first kind $T_n(\hat{x})$, such that $TB_n(\hat{x}) \equiv T_n(\hat{x})$, where $\hat{x}$ is given in terms of $\hat{\chi}$ as follows [e.g., Boyd, 2001]:

$$\hat{x} = \frac{\chi}{\sqrt{\rho + \chi^2}}$$

(27)

where $l > 0$ is a mapping parameter. Finding an optimal mapping parameter is crucial as it allows for a function to be represented with a minimal number of terms in the series, thus making simulation efficient. In Appendix B, we discuss how we identify the optimal $l$ in our calculations.

All $TB_n(\hat{x})$ converge toward unity as $|\hat{x}| \to \infty$; particularly $TB_0(\hat{x}) = 1$. Clearly, for all $m$, $G_m(\hat{x})$ and hence $\Delta\hat{\rho}_m(\hat{x})$ decay as $|\hat{x}| \to \infty$ (consistent with the pressure condition at infinity). Integrating $G_{2m}(\hat{x})$ from 0 to $\infty$ yields $-m\pi/2$. At $\hat{x} = 0$, we obtain $G_{2(2m-1)}(0) = -2$ and $G_{4m}(0) = 0$; in particular $G_2(0) = -2$ and $G_4(0) = 0$. Finally, we add $G_4(\hat{x})$ to $G_{2(2m+1)}(\hat{x})$ and $G_2(\hat{x})$ to $G_{4m+1}(\hat{x})$ with appropriate weighting to form the final basis set as expressed in equations (24) and (25), respectively, so that (i) each basis has a value of unity at $\hat{x} = 0$ (consistent with the pressure condition at the point of injection) and (ii) integration of each basis function from 0 to $\infty$ yields zero (consistent with the criterion that the resultant pressure must vanish).

Basis functions defined above, however, do not satisfy the pressure gradient inequality at the point of injection (i.e., equation (18)). As in Spence and Sharp [1985] and Adachi and Detournay [2002], we include a particular function

$$\Delta\hat{\rho}_n(\hat{x}) = \frac{1}{l} \{ \exp^{-|\hat{x}|(l - |\hat{x}|)} \}$$

(28)

in the series expansion that satisfies equation (18) as well as all the conditions and criteria discussed above. Note that the particular solution chosen is the “mapped” Laguerre basis function (associated with the Laguerre polynomial of order 1) for the semi-infinite interval. Mapping is consistent with that of other basis functions (i.e., equations (24) and (25)) in that the amplitudes of all these pressure terms are insensitive to $l$.

The first few basis functions, for example, are plotted in Figure 2a for $l = 1$.

Since the basis functions individually satisfy the pressure boundary conditions (both at $\hat{x} = 0$ and as $|\hat{x}| \to \infty$) as well as the zero resultant pressure criterion, $\Delta\hat{\rho}(\hat{x}, \hat{t})$ also satisfies all of these conditions/criteria provided that

$$D = \sum_{n=0}^{2N} a_n.$$  

(29)

In addition, $\Delta\hat{\rho}(\hat{x}, \hat{t})$ also satisfies the remaining criterion of pressure gradient inequality at $\hat{x} = 0$ via $\Delta\hat{\rho}_0(\hat{x})$.

With a series solution of $\Delta\hat{\rho}(\hat{x}, \hat{t})$, we can also express the solution for $\Delta\hat{\psi}(\hat{x}, \hat{t})$ as the following series:

$$\Delta\hat{\psi}(\hat{x}, \hat{t}) = \frac{1}{D} \sum_{n=0}^{2N} a_n \Delta\hat{\psi}_n(\hat{x}),$$

(30)

where each pair of pressure and displacement terms, i.e., $\Delta\hat{\rho}_n$ and $\Delta\hat{\psi}_n$, individually satisfy the elasticity equation (equation (13)). The first few displacement terms are plotted in Figure 2b for $l = 1$. We again note that the analytic evaluation of $\Delta\hat{\psi}_n$ is possible due to the appropriate combination of rational Chebyshev functions while defining the corresponding $\Delta\hat{\rho}_n$ (equations (24) and (25)).

Coefficients $a_n$ appearing in equations (23) and (30) are then obtained by satisfying the remaining governing equation, i.e., the fluid flow equation (equation (14)). The LHS and RHS of the equation can be expressed in the following series form:

$$\text{LHS} = \frac{\xi}{\Delta t} \left\{ \left[ \frac{1}{D} \sum_{n=0}^{2N} a_n \int_{\hat{x}}^{\infty} \Delta\hat{\psi}_n(\hat{x}) \, d\hat{x} \right] - \left[ \frac{1}{D} \sum_{n=0}^{2N} a_n \int_{0}^{\infty} \Delta\hat{\psi}_n(\hat{x}) \, d\hat{x} \right] \right\},$$

(31)

$$\text{RHS} = -\frac{2\chi}{l^{1/2}} \left[ \frac{\xi}{\Delta t} \left[ \hat{\psi} + \frac{1}{D} \sum_{n=0}^{2N} a_n \Delta\hat{\psi}_n \right] \right]^{5/3} \left[ \frac{\chi}{D} \sum_{n=0}^{2N} a_n \frac{\partial \Delta\hat{\rho}_n}{\partial \hat{x}} \right]^{1/2},$$

(32)

where $\Delta \hat{t}$ is the simulation time step. For the first time step of the simulation, the second term on the RHS of equation (31) is essentially zero due to the initial condition, i.e., $\Delta\hat{\psi}(\hat{x}, 0) = 0$ (cf. Figure 1b and equation (3)).
Figure 2. First few terms appearing in the series. (a) Pressure terms that are defined by the modified rational Chebyshev functions (equations (24) and (25)). Each term satisfies the pressure boundary conditions at \( \hat{x} = 0 \) and as \( \hat{x} \to \infty \), as well as the criterion defined by equation (17). A particular function (equation (28)) is also included in the series, \( \Delta \hat{p}_0 \), in order to satisfy the additional condition given by equation (18). (b) Displacement terms that are obtained from the elastostatic equation. For a given pressure term, the RHS of equation (13) is evaluated in a Cauchy principal value sense. The whole of equation (13) is then integrated from arbitrary \( \hat{x} \) to \( \infty \), satisfying the displacement boundary condition as \( \hat{x} \to \infty \). (c) Gradient of pressure terms. Notice that only \( \Delta \hat{p}_0 \) has a nonzero pressure gradient at \( \hat{x} = 0 \). (d) Integration of displacement terms from arbitrary \( \hat{x} \) to \( \infty \). All these terms are evaluated analytically for the choice of mapping parameter \( l = 1 \) (cf. equation (27)). Note that the amplitudes of pressure terms are insensitive to \( l \). This, however, is not true for displacement terms.

Note that the time discretization employed in equation (31) is a classical backwards Euler approach. Pressure gradient \( \Delta \hat{p}_j / \hat{a} \hat{x} \) and integral \( \int_0^\infty \Delta \hat{w}_j \, d\hat{x} \) appearing in the above equations can also be evaluated analytically. The first few terms for \( l = 1 \) are plotted in Figures 2c and 2d, respectively.

We finally calculate \( a_n \) at each given time \( \hat{t} \) by minimizing the following objective function (in a least squares sense) over a sufficiently large number of Gauss-Chebyshev quadrature points:

\[
R^2 = \min_{a_n} \sum_{j=1}^J \left\{ \text{LHS} \left( \hat{x}_j, \hat{t}, \hat{t} - \Delta \hat{t} \right) - \text{RHS} \left( \hat{x}_j, \hat{t} \right) \right\}^2,
\]

where \( R^2 \) is the squared 2-norm of the residual, LHS and RHS are explicitly given by equations (31) and (32), and \( \hat{x}_j \) are the quadrature points and given by

\[
\hat{x}_j = \frac{I x_j}{\sqrt{1 - x_j^2}}; \quad \hat{x}_j = \cos \left( \frac{2j - 1}{4J} \pi \right); \quad j = 1, 2, \ldots, J.
\]

While optimizing equation (33), we constrain \( \hat{w}(\hat{x}, \hat{t}) \) to be positive, i.e., \( \hat{w} \geq e \), where \( e \) is a small positive number that ensures the continuity of opening during the flood propagation (cf. section 4.1). We employ the constrained nonlinear optimization function \texttt{fmincon} of MATLAB® with the \texttt{sqp} algorithm [e.g., Nocedal and Wright, 2006]. The global minimum is found using the \texttt{multistart} solver of the MATLAB® Global Optimization Toolbox.

We choose appropriate values of \( J, N \), and \( l \) for our simulations (cf. Appendix B). For convenience, we use a variable time step \( \Delta \hat{t} \) that corresponds to 2.5, 5, and 10 min of drainage; smaller steps are taken at early times. Chosen time steps are small enough to yield stable solutions and large enough to simulate the entire drainage events efficiently.
4. Results

For the governing equations that are nondimensionalized as in section 3.1, solutions depend on the choice of initial opening, $w_i$. The hypothesized preexisting opening may characterize a thin sheet of water [e.g., Weertman, 1972; Le Brocq et al., 2009] potentially within a distributed system of linked cavities [e.g., Walder, 1986; Kamb, 1987]. It is virtually impossible to measure the height of the subglacial openings, their spatial distribution, and temporal variations. Cavities are generally larger, with heights maybe on the order of a few centimeters [e.g., Kamb, 1987; Schoof et al., 2012]. Orifices that connect the distributed cavities are relatively smaller; these may have average heights of a few millimeters [e.g., Kamb, 1987; Fountain and Walder, 1998]. Since the slow drainage system of linked cavities transmits water efficiently at high pressure [e.g., Fountain and Walder, 1998; Schoof, 2010], ice slides faster in such cases. The associated regelation-sliding mechanism may thus produce microscopic films of water even in the regions that are not connected to the drainage network [e.g., Nye, 1973].

Apart from large uncertainties associated with the initial opening height, it is also not known a priori what type of drainage system (cavities or microscopic water films) exists underneath the prospective drainage sites. We therefore investigate three plausible scenarios, with $w_i = 10$, 1, and 0.1 cm. This hypothesized preexisting subglacial opening may be interpreted as the average height of a sheet/film of water within a distributed system of linked cavities as in, for example, Hewitt [2011] and Werder et al. [2013].

4.1. Along-Flow Profiles of the Field Variables

In this section, we evaluate how profiles of field variables evolve as foods propagate. We first describe one example scenario, with $w_i = 10$ cm; these solutions have all of the elements present in corresponding solutions for other scenarios (to be discussed in section 4.3). Along-flow profiles of excess pressure, normalized by the pressure at the point of injection, are shown in Figure 3a. Corresponding profiles of normalized total elastostatic opening (Figure 3b), fluid flux (Figure 3c), and depth-averaged fluid velocity (Figure 3d) are also plotted. Solutions are shown only for five representative times.
General characteristics of the profiles of each variable during the entire flooding event are similar. As Figure 3a depicts, $\Delta \hat{p}$ decreases from its maximum value at $\hat{x} = 0$, becomes negative, attains a minimum value, and then increases asymptotically toward zero, thus giving zero resultant in order for displacement to remain finite as $\hat{x} \to \infty$ (cf. section 2.3). From a physical point of view, importantly, there exists a positive gradient in $\Delta \hat{p}$ beyond some distance from the point of injection. This reversal of pressure gradient draws water in from the far field, thus requiring $\hat{q}$ (and $\hat{v}$) to be negative beyond some $\hat{x}$. In other words, the direction of fluid flow reverses in the far field. This may not be evident from Figure 3c as the magnitudes of negative flux are much smaller than the large positive fluxes closer to the point of injection (cf. section 4.2) but is apparent in Figure 3d where negative velocities exist in regions of positive pressure gradient.

Reversal of water flow also implies that the total opening must narrow, respecting mass continuity. In other words, $\hat{w}_e$ must be negative beyond some $\hat{x}$. This is evident in Figure 3b, particularly in the late stages of flooding. As noted in section 3.2, the limit for negative $\hat{w}_e$ is restricted to $(-\hat{w}_e + \epsilon)$, where $\epsilon$ is a small positive number, in order to ensure the existence of ice/bed opening. It is important to stress here that a hypothesis intrinsic to our model is the continuation of opening during the entire flooding event (i.e., the subglacial opening must not cease to exist). In reality, lateral variations in bed geometry may prohibit the total ice/bed opening from complete closure. (See section 6 for more discussions.)

We experiment with different values of $\epsilon$ and find that the choice of $\epsilon$ does not affect the solutions as long as $\epsilon \ll \hat{w}_e$. We choose $\epsilon = 10^{-10}$ for all of our calculations. In the early stages of flooding when the minimal ice/bed opening is much larger than $\epsilon$, we notice asymptotic approach of field variables (i.e., $\hat{w}_e$, $\hat{q}$, and $\hat{v}$) toward zero in the far field. This is particularly evident in the velocity profiles (Figure 3d). (Minimal total opening does not narrow to the order of $\epsilon$ until 80 min of SGL drainage; cf. section 4.3.) Once the total opening becomes too narrow (on the order of $\epsilon$) in the late stages of flooding, field variables do not monotonically approach zero in the far field as $\hat{x} \to \infty$. Instead, there are small random variations in the field variables (see, e.g., Figure 3d). These variations, however, do not affect the overall solutions. Displacement profiles, for example, have similar shapes irrespective of the magnitude of minimal total opening, and thus irrespective of the absence/presence of small variations in field variables in the far field. We believe these variations arise partly from optimization errors, which are minimized by considering a large number of terms in series (cf. Appendix B). Understanding the behavior of field variables in these microregions might be useful but is not crucial to the scope of this study given its negligible impact on general profiles of field variables as noted earlier.

To translate our nondimensional results into dimensionally meaningful ones, based on the choice of values listed in Table 1, we note that $\hat{x} = 1$ corresponds to $x = x_0 \approx 4.77$ km, $\hat{w}_e = 1$ to $w_e = w_0 \approx 0.61$ m, $\hat{q} = 1$ to $q = q_0 \approx 0.40$ m$^2$ s$^{-1}$, and $\hat{v} = 1$ to $v = v_0 \approx 0.66$ m s$^{-1}$.

4.2. Important Attributes of the Solutions

Here we analyze some important attributes of the solutions presented in section 4.1 and discuss their physical significance. Maxima and minima of field variables and characteristic widths of the solutions are extracted. Although the analysis is based on one particular scenario, i.e., for $w_i = 10$ cm, the same conclusions generally hold for the other scenarios as well (cf. section 4.3).

As part of the boundary conditions (cf. section 2.3), $\Delta \hat{p}$ is always equal to unity at the point of injection, and pressure profiles (Figure 3a) show that this is also the maximum value. Maxima of other field variables are plotted in Figure 4a. (Note that these all occur at $\hat{x} = 0$; see Figure 3.) Maxima of $\hat{w}_e$ and $\hat{q}$ both evolve following near-linear trends, with $\hat{q}$ being roughly 10 times $\hat{w}_e$, but the rate of increment in the maximum $\hat{q}$ is slightly higher in the late stages of flooding. Thus, the maximum value of $\hat{v}$ increases gradually as the flood propagates.

We also extract the minima of the field variables. These are all negative during the entire flooding event. In Figure 4b, we plot the absolute values of the minima of some field variables with respect to the corresponding maxima. In other words, we plot amplitude ratios in order to quantify the relative magnitudes of the limiting values. The absolute amplitude ratio for $\Delta \hat{p}$ increases rapidly in early stage of flooding, until it finds a more or less steady value. After 2 h of drainage, for example, the minimum $\Delta \hat{p}$ is $\approx 50\%$ smaller in magnitude than the imposed pressure at $\hat{x} = 0$. Note that the evolution of the amplitude ratio for $\Delta \hat{p}$ characterizes that of the pressure minima because the maximum $\Delta \hat{p}$ is fixed in time.
Figure 4. Some important attributes of the solutions presented in Figure 3 (i.e., for \( w_i = 10 \) cm). (a) Maximum values of the field variables (that occur at \( \hat{x} = 0 \)). Note that results for \( \hat{q} \) are contracted by a factor of 10 for ease of comparison. (b) Absolute amplitude ratios, defined as the ratio between the minima and maxima of the field variables. For clarity, solutions for \( \hat{w}_e \) and \( \hat{q} \) are exaggerated by a factor of 10. (c) Normalized distance at which the variables first become negative. During the entire drainage event, \( \Delta \hat{p} \) changes sign relatively close to the point of injection, while \( \hat{w}_e \) does so at the farthest distance. (d) Locations at which the variables attain their minimum values. Relative locations for the field variables are similar as in Figure 4c, but the minima occur within a much narrower range of \( \hat{x} \).

The amplitude ratio for \( \hat{w}_e \) is essentially the ratio between maximum subsidence in the far field and maximum uplift that occurs at the point of injection. Maximum subsidence increases gradually until it reaches the limiting value defined by the initial opening. (This limiting value is reached only after 80 min of drainage; cf. section 4.3.) The evolution of the amplitude ratio for \( \hat{w}_e \) characterizes that of both subsidence and uplift in the early stage of flooding, but solely that of uplift in the late stages. Thus, a systematic reduction in amplitude is evident after 80 min of flooding. In summary, maximum subsidence is only about 1–2% of the maximum uplift during the entire event.

Similarly, the amplitude ratio for \( \hat{q} \) quantifies the magnitude of flux reversal in the far field relative to the maximum flux observed at the point of injection. We find no apparent trend in the evolution of the maximum reversed flux (results not shown), but fortunately, their magnitudes are much smaller. Evolution of the amplitude ratio for \( \hat{q} \) is thus dominated by that of maximum \( \hat{q} \), particularly in the middle and late stages of flooding. During these periods, the maximum reversed flux is on the order of 0.1% of the maximum flux.

In order to evaluate how far the system is perturbed during the drainage event, we consider two characteristic widths of the solutions, namely, the distance at which variables change sign (Figure 4c) and the distance at which variables attain their minima (Figure 4d). Figures 4c and 4d reveal that the characteristics widths evolve more or less linearly for all field variables. During the entire drainage event, \( \Delta \hat{p} \) changes sign relatively close to the point of injection, and \( \hat{w}_e \) does so at the farthest distance; corresponding locations for \( \hat{q} \) fall in between, but closer to the latter limit (cf. Figure 4c). The same is generally true about the relative locations for field variables where they attain minima (Figure 4d). In this case, however, solutions for \( \hat{q} \) are closer to those for \( \Delta \hat{p} \). It is also evident from Figures 4c and 4d that the minima occur within a much narrower range of \( \hat{x} \) compared with the range for the changes in sign.

Note that the location at which \( \hat{q} \) changes sign (Figure 4c) is equivalent to the location where \( \Delta \hat{p} \) is minimum (Figure 4d). We may term this as the “flood front” in that it is the location beyond which flux reverses.
4.3. Effects of the Initial Opening

The explanations presented in sections 4.1 and 4.2 generally apply to the other two scenarios as well, i.e., for \( w_i = 1 \) and 0.1 cm. In this section, we compare the results for different choices of \( w_i \) and highlight the differences as appropriate. We specifically evaluate the effect of \( w_i \) on along-flow profiles (Figure 5), limiting values of field variables (Figure 6), and characteristic widths of the solutions (Figure 7). (Note that we also plot hydrofracture solutions of Tsai and Rice [2010] in some of the figures but do not comment on these until section 5.)

Effects of initial opening on the evolution of along-flow profiles of field variables are shown in Figure 5. Comparisons are made for three representative times. Although corresponding profiles (i.e., profiles for the given variable at the given time) appear to have similar shape irrespective of \( w_i \), there are systematic differences among the solutions. Greater perturbations (both higher absolute magnitudes and larger solution widths) are found for larger \( w_i \). Results also suggest that differences in corresponding solutions for greater initial openings (i.e., between \( w_i = 10 \) and 1 cm) are much larger than those for narrow openings (i.e., between \( w_i = 1 \) and 0.1 cm).

Even during early stages of flooding, small variations in field variables are present in the far field for cases with narrow \( w_i \). These are evident particularly from the velocity profiles (Figure 5d). The variations, as discussed in section 4.1, are mainly associated with narrow opening in the far field. The minimal total opening narrows to the order of \( \varepsilon \) (or, in other words, maximum subsidence reaches the limit defined by \( w_i \)) after 80 and 20 min of drainage, respectively, for \( w_i = 10 \) and 1 cm, but it happens from the very early stage (at least, 2.5 min) of drainage for \( w_i = 0.1 \) cm. See Figure 6a where the minima of total elastostatic opening are plotted. Uniform magnitudes imply that minimal total opening narrows down to the order of \( \varepsilon \). As reported earlier, this happens faster for smaller \( w_i \). It is important here to recall from section 4.1 that minor variations appearing in the far field do not alter the overall profiles of corresponding field variables.

Figure 6a also compares the evolution of the pressure minima. Irrespective of \( w_i \), we find that the minimum \( \Delta p \) decreases rapidly in the early stages of flooding and reaches relatively stable values thereafter. Interestingly, the pressure minima are found to be less negative for wider \( w_i \) until steady values are reached after
Figure 6. Effects of initial opening on magnitudes of field variables. (a) Minimum values of normalized excess pressure and normalized total elastostatic displacement. For ease of comparison, solutions for \( \hat{w} \) are exaggerated by a factor of \( c = 100 \) (for \( w_i = 1 \text{ mm} \)) and 10 (\( w_i = 1 \text{ cm} \)). Maximum values of normalized (b) total elastostatic opening, (c) fluid flux, and (d) depth-averaged fluid velocity in different flooding scenarios. For each field variable at a given time, higher maximum values are found for larger \( w_i \). Note that all maxima occur at \( \hat{x} = 0 \). Hydrofracture solutions of Tsai and Rice [2010] are also plotted as appropriate.

Figure 7. Effects of initial opening on the characteristic width of the solutions. (a) Normalized distance where excess pressure and total elastostatic displacement change sign. (b) Normalized distance where fluid flux changes sign (equivalently, location of the flood front). Hydrofracture solutions of Tsai and Rice [2010] are also plotted as appropriate. (c) Fraction of \( \hat{x} (\hat{w} = 0) \) where excess pressure and fluid flux change sign. In general, field variables change sign within a narrow range of relative distance for larger \( w_i \). (d) Relative location of the flood front plotted against \( w_i \). Solutions are more sensitive to \( w_i \) at early times of flooding.
\( \approx 30 \) min of drainage, after which the situation seems to reverse although systematic differences in the solutions are not apparent.

We also compare maxima of field variables, which always occur at \( \tilde{x} = 0 \), for different choices of \( w_i \) (Figures 6b–6d). Results suggest that maxima of each variable are systematically higher for larger \( w_i \) during the entire drainage events. Differences in corresponding solutions for greater initial openings (i.e., between \( w_i = 10 \) and \( 1 \) cm) are, once again, much larger than those for narrow openings (i.e., between \( w_i = 1 \) and \( 0.1 \) cm). For all \( w_i \), we find that the maxima of \( \tilde{\nu}_c \) (Figure 6b) and \( \tilde{q} \) (Figure 6c) evolve near linearly. Maxima of \( \tilde{\nu} \) also increase with time, but gradually, particularly for larger \( w_i \) (Figure 6d). Velocity solutions approaching each other in the late stages of flooding imply that the role of \( w_i \) diminishes as floods propagate. The same conclusion can be reached from other solutions by presenting them appropriately (results not shown).

In order to evaluate the effect of \( w_i \) on the characteristic width of the solutions, we compare corresponding locations where variables change sign. For clarity, solutions for \( \tilde{\nu}_c \) and \( \Delta \tilde{\rho} \) are plotted in Figure 7a and those for \( \tilde{q} \) in Figure 7b. (Recall from section 4.2 that the location where \( \tilde{q} \) changes sign is termed the flood front.) For each variable, characteristic widths of the solutions generally increase with \( w_i \) during the entire flooding events. Much larger differences in solution width are found between \( w_i = 10 \) and \( 1 \) cm than between \( w_i = 1 \) and \( 0.1 \) cm. Irrespective of \( w_i \), we find that \( \tilde{\nu}_c \) changes sign at the farthest distance (relative to the locations where \( \tilde{q} \) and \( \Delta \tilde{\rho} \) change sign) from the point of injection (see, e.g., Figure 4c). In Figure 7c, we plot characteristic widths for \( \tilde{q} \) and \( \Delta \tilde{\rho} \) relative to those for \( \tilde{\nu}_c \). Results suggest that relative locations are roughly constant during the entire drainage events, except possibly for \( w_i = 10 \) cm in early stage; \( \tilde{q} \) and \( \Delta \tilde{\rho} \) change sign \( \approx 5–15\% \) and \( 35–45\% \) closer to the point of injection, respectively. Lower limits are generally associated with larger \( w_i \), implying that variables in such cases change sign over a narrow range of relative distance.

Finally, we highlight the effect of \( w_i \) on the rate of flood propagation (Figure 7d). Figure 7d shows that floods travel farther for larger \( w_i \) during the entire drainage events. Moreover, the rate of flood propagation increases much more rapidly with \( w_i \) (compare the steep gradients between \( w_i = 10 \) and \( 1 \) cm versus the relatively flat ones between, say, \( w_i = 0.1 \) and \( 0.01 \) cm). However, the effect of \( w_i \) diminishes as floods propagate. Relative to locations of the flood front for \( w_i = 10 \) cm, the flood travels only \( \approx 65\% \) of the distance in a scenario with \( w_i = 0.1 \) cm after \( 10 \) min of drainage, but it covers \( \approx 88\% \) of the distance after \( 2 \) h.

Based on the results presented above, we may summarize our findings that perturbations in the subglacial hydrological system grow increasingly more rapidly with increasing \( w_i \), but the role of \( w_i \) tends to diminish as floods propagate.

5. Comparison With a Hydrofracture Model

Our model cannot evaluate the propagation of a subglacial flood in fully coupled ice/bed scenarios (effectively, \( \tilde{\nu}_c = 0 \)), as our model requires the continuation of the opening (i.e., \( \tilde{\nu}_c > 0 \)) over sufficiently long distances during the entire drainage event. In such fully coupled situations, floods may propagate through hydrofracturing by opening the crack along the ice/bed interface [e.g., Tsai and Rice, 2010, 2012]. It is nevertheless useful to compare our solutions, particularly as \( \tilde{\nu}_c \to 0 \), with corresponding hydrofracture solutions. Prior to that, we briefly introduce the turbulent hydrofracture model of Tsai and Rice [2010] and highlight fundamental elements on which this model and our model differ. It is the most relevant hydrofracture model in the present context because, as we do, Tsai and Rice [2010] treat ice and bed as purely elastic and consider that the subglacial flood is fully turbulent. In addition, free surface effects [Tsai and Rice, 2012] are neglected in both models.

One of the key differences lies in the elasticity equation, which relates the applied external pressure to elastostatic opening. The elasticity equation in our model is based on a half-plane problem (equation (1)), and the total elastostatic opening is given by the displacements of two half planes represented by the ice and bed (cf. Appendix A). On the other hand, the elasticity equation applied in hydrofracture models is based on a plane problem with a finite crack in it. For a homogeneous ice body, we may write [e.g., Muskhelishvili, 1953]

\[
\frac{\partial w_c(x, t)}{\partial x} = \frac{4}{\pi E} \int_{-L}^{L} \frac{\Delta p(s, t) \, ds}{s-x},
\]

where \( w_c(x, t) \) is the full opening of the crack at time \( t \) and \( L(t) \) is the corresponding crack half length (cf. Figure 1a). Here we emphasize that the hydrofracture model domain is finite, such that \( x \in [-L, L] \), in
contrast to our infinite domain. As a result, equation (35) can be inverted to evaluate the distribution of excess pressure, given the displacement profile [e.g., Sneddon and Lowengrub, 1969].

Because the crack is located in a plane along the interface of dissimilar materials, the actual elastostatic opening, and thus the total crack opening, is given by

\[ w(x, t) = w_c(x, t) = \zeta w_c(x, t), \]

where \( w_c \) is the full opening of the crack for a homogeneous plane of more compliant material and \( \zeta \approx 0.55 \) accounts for the bimaterial effect (from Tsai and Rice [2010], based on the material parameters listed in Table 1). Note that \( \zeta = 0.5 \) represents the case with rigid bed, whereas \( \zeta = 1 \) characterizes the situation with an englacial crack.

The fluid flow equation is similar to equation (8) and is as follows:

\[ \zeta \frac{\partial}{\partial t} \int_0^L w_c(x, t) \, dx = 2 \left[ \frac{1}{L_0 \rho k^{1/3}} \right] \frac{1}{2} \left( -\frac{\partial \Delta p(x, t)}{\partial x} \right)^{1/2}. \]  (37)

Note that, unlike in our model, the reversal in the fluid flow does not occur in this case. Also, note that the equation presented is only applied to the right half of crack. Due to the problem symmetry, we again focus on this half domain only.

Equations (35) and (37) form the governing equations for the hydrofracture model of Tsai and Rice [2010]. Boundary conditions and criteria presented in section 2.3 hold for this model as well, except that (i) the criterion analogous to equation (9) is automatically satisfied for a crack pressurized by fluid, (ii) far field limits should be at \(|x| = L\) instead of \(|x| \to \infty\), and (iii) the pressure is not defined in the far field, but is a part of the solutions. (It is well known that there exists pressure singularities at the crack tips, i.e., \( \Delta p(|x| \to L, t) \to -\infty \). Desroches et al. [1994], however, show that these are weaker than classical elastic square root singularities.) Instead, we must satisfy the following fracture criterion at the crack tips:

\[ K_I \equiv \sqrt{\pi L} \int_0^L \Delta p(x, t) \sqrt{L^2 - x^2} \, dx = K_{Ic} \approx 0, \]  (38)

where \( K_I \) is the mode I stress intensity factor and \( K_{Ic} \) is the material toughness that is assumed to be negligible. Furthermore, Tsai and Rice [2010] assume that there exists zero lag between the crack tip and flood front during the entire drainage event.

Through appropriate scaling, it can be shown that a self-similar solution exists for typical hydrofracture problems [e.g., Spence and Sharp, 1985; Adachi and Detournay, 2002]. Tsai and Rice [2010] present self-similar solutions that incorporate crack tip asymptotes for pressure and elastostatic opening [Desroches et al., 1994]. We express these solutions in our scaling as follows:

\[ \hat{x} = \frac{1}{x_0} \left( L(t) \hat{x}^{ss} \right), \]  (39)

\[ \hat{\Delta p} = \frac{1}{\Delta p_0} \left( \Delta p_0 \hat{\Delta p}^{ss} \right), \]  (40)

\[ \hat{w}_c = \frac{1}{w_0} \left( \zeta L(t) \frac{\Delta p_0}{E} \hat{w}_c^{ss} \right), \]  (41)

where

\[ L(t) = 5.77 \left[ \frac{\Delta p_0^7}{\rho^3 k E^4} \right]^{1/5} t^{6/5}. \]  (42)

Hatted variables with superscript "ss" denote self-similar solutions normalized as in Tsai and Rice [2010]. Terms enclosed by braces are corresponding solutions in real dimensions, which are again normalized as in section 3.1 in order to compare these with our results. (We refer to them as hydrofracture solutions.) Note that \( \hat{x}^{ss} \in [0, 1] \).
Figure 8. Summary of comparison of our results with the hydrofracture solutions of Tsai and Rice [2010]. (a) Total elastostatic opening at the point of injection and (b) location of the flood front, with respect to corresponding self-similar, hydrofracture solutions. Greater discrepancies are found at early times and for larger \( w_i \).

We plot hydrofracture solutions for pressure and displacement profiles in Figure 5. Crack tip singularities are evident in the \( \Delta \hat{p} \) profiles. Normalized distances where \( \Delta \hat{p} \) changes sign are \( \approx 0.85 L(t)/x_0 \), which appear shorter than corresponding distances for our results. Displacement profiles suggest that the magnitudes of \( \hat{w}_e \) are always smaller in hydrofracture solutions, although the profile shapes are all similar to our results except near the crack tips. Note that the minima of the elastostatic opening are always zero in the hydrofracture solutions and occur at the crack tips as a part of the boundary condition.

We extract some important attributes of the hydrofracture solutions and compare with our results as appropriate. Evolution of the maximum \( \hat{w}_e \) is shown in Figure 6b. Locations at which \( \Delta \hat{p} \) changes sign during the drainage event are compared in Figure 7a. Similarly, propagation of the flood front is also compared (Figure 7b). (Note that the location of the flood front in hydrofracture solutions is the same as the crack half length.) These comparisons suggest that hydrofracture solutions are smaller than corresponding solutions of our model; narrow opening and small rate of flood propagation are particularly evident during the drainage events. It is consistent with a prior conclusion (i.e., perturbations in the subglacial system increase with \( w_i \)) in that our solutions systematically approach the hydrofracture solutions (effectively, \( w_i = 0 \) cm) as we narrow \( w_i \). However, discrepancies between the hydrofracture and our solutions appear to be larger than anticipated.

We do not fully understand why such large discrepancies exist between the model solutions. One possible reason could be related to the self-similar nature of the hydrofracture solution, which does not depend on initial conditions. This is evident from Figure 8 where we plot characteristic opening and rate of flood propagation relative to the corresponding hydrofracture solutions. Solution discrepancies are large in early times but decrease rapidly as floods propagate. Maximum \( \hat{w}_e \) for \( w_i = 10 \) cm, for example, is about 2.87 times wider after 10 min but is only 1.39 times wider after 2 h of drainage. Similarly, floods propagate more than 300% faster in early times and only 68% faster after 2 hours for the same scenario. Another reason for the large discrepancies between the hydrofracture and our model solutions could be related to the concentration of stress in front of the crack tips [e.g., Anderson, 2005]. The infinite stress concentration at the crack tips, which is predicted by LEFM but does not occur in reality, may impede the pace of flood propagation during the hydrofracture model simulation even though negligible \( K_I \) is assumed. We will discuss this more in section 6.

6. Discussion

In the theory presented, we solve for subglacial flooding through a preexisting film of water during the rapid drainage of SGLs. The consideration of a preexisting water film is generally consistent with the spirit of contemporary continuum models for subglacial hydrology [e.g., Flowers and Clarke, 2002; Le Brocq et al., 2009; Schoof, 2010; Hewitt, 2011; Schoof et al., 2012; Hewitt et al., 2012; Werder et al., 2013; de Fleurian et al., 2014]. Nevertheless, applicability of our model relies on how well the hypothesis of a preexisting water film of uniform thickness characterizes the real subglacial hydrological conditions underneath the prospective drainage site. The plane strain representation of our model requires that the extent of an initial water film must be sufficiently large in both horizontal dimensions. In addition, the thin film of water may narrow in
the far field as a consequence of reversal in pressure gradients, but it must not cease to exist for the theory to hold during the entire drainage event.

In reality, the subglacial hydrological network may be comprised of morphologically distinct systems of water storage and pathways for drainage (see, e.g., *Fountain and Walder* [1998] for a review). It may include well-developed Röthlisberger channels [e.g., *Röthlisberger*, 1972; *Lliboutry*, 1983], networks of linked cavities [e.g., *Walder*, 1986; *Fowler*, 1987; *Kamb*, 1987], widespread thin sheets (thickness of order 1 mm) of basally produced (e.g., geothermally heated) water [e.g., *Weertman*, 1972; *Walder*, 1982; *Creyts and Schoof*, 2009], and interfacial films of water associated with the regelation-sliding process (thickness of order 1 μm [e.g., *Nye*, 1973]) or contained on premelted ice (thickness of order 1 nm; e.g. *Dash et al.* [2006]). The best analogy of our model framework (i.e., presence of an initial water film) would be that of a slow drainage system, ideally characterized by a widespread sheet of water [e.g., *Weertman*, 1972; *Walder*, 1982] potentially within a distributed network of linked cavities [e.g., *Walder*, 1986; *Kamb*, 1987]. The sheet-like flow is also hypothesized in a number of subglacial hydrological models [e.g., *Creyts and Schoof*, 2009; *Le Brocq et al.*, 2009] and supported by several field studies as well [e.g., *Jóhannesson*, 2002; *Magnússon et al.*, 2007]. The distribution of rock clasts, asperities and large bumps [e.g., *Kamb*, 1987; *Creyts and Schoof*, 2009] can control the instability inherent to a sheet-like flow [e.g., *Walder*, 1982]. It may also potentially prevent the complete closure of an initial opening along the ice/bed interface during suction. When the base of the ice vertically approaches the bed, a microscopic layer of water may likely be trapped or produced locally (due to, e.g., frictional heating and geothermal flux), and such a thin film of water is sufficient for our theory to hold as numerically demonstrated in section 4. So, the hypothesis of the continuation of opening between the ice and its bed may be generally valid. To summarize, our model applies, at a minimum, to warm-based interior ablation regions of ice sheets, where an efficient (channelized) drainage system is absent (or not fully developed) and where instead a widespread thin film of interfacial water, if not a relatively thick sheet of water within a system of linked cavities, is expected to exist underneath the prospective drainage site.

There are several other assumptions, perhaps of lower order importance, associated with our model. We consider high Reynolds number flow (i.e., fully turbulent), assume a lubrication theory for fluid flow (i.e., 1-D flow along the x axis [e.g., *Bird et al.*, 1987]) with neglect of the acceleration term from the Navier-Stokes equation, ignore sloping bed topography, and assume a time invariant excess pressure at the point of injection during the entire drainage events. These assumptions are generally valid for typical rapid drainage events on the GrIS [e.g., *Das et al.*, 2008], as thoroughly discussed by *Tsai and Rice* [2010]. We also hypothesize that the initial opening is fully filled with water and that the water is at rest. These are reasonable assumptions as well, particularly in light of the interpretation that the preexisting thin film of water best characterizes a slow drainage system, which is known to transport water at high pressure [e.g., *Fountain and Walder*, 1998; *Schoof*, 2010]. Furthermore, our model neglects free surface effects as the elasticity equation (equation (1)) strictly corresponds to the half-plane problems. It essentially implies that the length scale over which the subglacial system is perturbed should be much smaller than the ice thickness itself. Unfortunately, this is not the case. Characteristic solution widths (cf., e.g., Figures 7a and 7b) are much larger than the normalized ice thickness, $H_{ice} = H_{ice} / x_0 \approx 0.21$ (Table 1), even in early times of flooding. Our model should therefore be improved by considering a more appropriate elasticity equation (than equation (1)) analogous to that given by, for example, *Erdogan et al.* [1973] for a crack located near surface. (*Tsai and Rice* [2012] include free surface effects to improve their original model, i.e., *Tsai and Rice* [2010]). We leave this important research for future work.

As noted earlier, the only available comprehensive model [*Tsai and Rice*, 2010, 2012] for subglacial flooding during the rapid drainage of SGLs hypothesizes that floods propagate through hydrofracturing. One may naturally consider it as a special case of our model when \( \dot{w} = 0 \), although our model does not hold in such a fully coupled ice/bed scenario. This seems to be a reasonable interpretation, particularly because our solutions systematically approach the hydrofracture solution as we narrow \( w_i \). Large discrepancies between our and hydrofracture model solutions, however, lead us to argue that there must be some fundamental differences, and hence, the hydrofracture model may not be a special case of our model. First, the hydrofracture solutions do not depend on the initial conditions as discussed in section 5. Second, and perhaps more importantly, the hydrofracture solutions are dictated by the infinite stress concentration at the crack tips that intrinsically arises from the employed LEFM approach. In reality, stresses at the crack tip must be finite in order to maintain a finite crack tip radius [e.g., *Anderson*, 2005]. A plastic zone may also form ahead of
the crack tip, leading to further relaxation of crack tip stresses. The available model does not capture these important processes associated with hydrofracturing that may effectively enhance the stress intensity factor [Dugdale, 1960; Barenblatt, 1962] and potentially reduce the discrepancies between the model solutions. However, a rigorous analysis is needed to fully understand this.

In this study, we do not attempt to model real drainage events [e.g., Das et al., 2008; Doyle et al., 2013; Tedesco et al., 2013]. Substantial effort would be needed to generalize our plane strain solutions to a more realistic 3-D (penny-shaped) geometry prior to simulating the SGL discharge rate. Similarly, appropriate methods should be identified to calculate the ice surface displacements (both vertical and horizontal) from the available solutions of subglacial elastostatic opening. Tsai and Rice [2010] employ reasonable ad hoc approaches to simulate the surface displacements and the rate of SGL water level drop observed by Das et al. [2008] during one particular drainage event. It is worthwhile to note that amplitudes of opening of the Tsai and Rice [2010] model were not large enough to reproduce the observations. Since our model does indeed have larger openings (by a factor of ≈1.5; Figure 8a), we speculate that our model might help to explain those original [Das et al., 2008] and similar observations. This important systematic comparison between model predictions and field measurements is left for future work.

Tsai and Rice [2010] thoroughly discuss the complications that my arise while modeling the real events. There is an additional complexity associated with our model as to the appropriate choice of an initial opening between the ice and the bed. However, certain field measurements such as lowering of subglacial water pressure or reversal in flow direction (these can be measured using a series of boreholes) and (minor) subsidence of the ice surface prior to the arrival of flooding, if available, may be utilized to directly constrain the initial opening. Alternatively, we may invert for the initial opening while fitting the other measurements such as surface uplift near the moulin and SGL drainage rate that are more readily available [e.g., Das et al., 2008; Doyle et al., 2013; Tedesco et al., 2013]. To be able to infer subglacial hydrological conditions in this way would be of paramount importance, because it would have direct constraints on hydrological models, basal sliding laws, and ice sheet dynamics in totality. For example, the inferred initial opening can be interpreted as the average height of a distributed system of linked cavities and used directly to constrain continuum hydrological models [e.g., Hewitt, 2011; Werder et al., 2013].

Given the strength and limitations discussed above, the theory and results presented in this study have at least three major implications. First, unlike the previous effort of Tsai and Rice [2010], our model is compatible with continuum subglacial hydrological models, because we model the flooding through a preexisting hydrological network and there are no solution singularities associated with our model. Second, our model can be potentially used to infer subglacial hydrological conditions from surface observables (e.g., SGL discharge rate and ice surface displacements). This information will be invaluable to constrain, for example, hydrological models and sliding laws. Third, our model has a wide scope of potential applications. It is obviously applied to the present-day GrIS, and its scope will only increase as more frequent drainage events are expected in the future [e.g., Sundal et al., 2009; Liang et al., 2012]. The model will be applicable to the Antarctic Ice Sheet as well, when climate warming eventually leads to the formation of SGLs in the future. Similarly, our model should also be useful to simulate the rapid collapse of the Laurentide Ice Sheet [e.g., Tarasov and Peltier, 1997] more accurately.

7. Conclusions

We present a model for subglacial flooding through a preexisting thin film of water during the rapid drainage of SGLs. We assume that the opening between the ice and the bed does not close completely during the entire drainage event; however, it can be as narrow as, for example, 1 μm representing a microscopic layer of water produced due to the regelation-sliding process. The hypothesized thin film of water, which can be interpreted as the average height of a distributed drainage system of linked cavities, is consistent with the spirit of continuum hydrological models. Using this framework also results in avoiding the solution singularities that are inherent to classical hydrofracture models, thus opening an avenue for integrating the likes of our (short-timescale) model within a computational suite of continuum models for subglacial hydrology.

We evaluate the effect of initial opening on the degree of perturbation to the subglacial system. We find that, irrespective of the initial opening, there exists a region of positive pressure gradient during the entire drainage event. This reversal of pressure gradient sucks water in from the far field and thus causes the
initial opening to narrow, respecting mass continuity. We find that along-flow profiles of corresponding field variables have similar shapes for all choices of initial opening, but greater perturbations (both amplitudes and wavelengths) are generally found for larger opening. For example, floods propagate much faster and displacements are much greater for larger initial openings.

Our solutions systematically approach the hydrofracture solution as we narrow the initial opening. However, there are still discrepancies between the corresponding model solutions, particularly in early times of flooding. This may be partly because of the self-similar nature of the hydrofracture solutions that does not depend on the initial conditions. Another plausible reason is that the hydrofracture solutions are dictated by unrealistic (infinite) stress concentrations in front of the crack tips, thus effectively reducing the stress intensity factor. Due to these fundamental differences, the hydrofracture model should not be mistaken as a special case of our model with zero initial opening.

Our model certainly has room for improvement. Capturing the free surface effect should be given priority. A rigorous effort is also needed to accurately parameterize the ice surface displacements and the rate of SGL discharge from available solutions of vertical displacement at the base. This would allow us to potentially invert for subglacial hydrological conditions, particularly the thickness of a preexisting water film, from surface observables. Thus, inferred subglacial conditions will have great significance in constraining hydrological models, basal sliding laws, and hence ice sheet dynamics. Our model also has a wide scope of potential applications. It is applied to the present-day GrIS, and its scope will only increase in an ongoing warming climate. It will be useful to modeling parts of the Antarctic Ice Sheet as well, when climate warming eventually leads to the formation of SGLs in the future. Our model is also relevant to paleo-icesheets. For example, it should complement the system of models to simulate the rapid collapse of the Laurentide Ice Sheet more accurately.

Appendix A: Bimaterial Approximation

On either side of the subglacial hydrological network, there are different materials, namely, the ice and bed (Figure 1b). The total elastostatic opening in section 2.2 is defined as $\zeta$ times the elastostatic opening of the (upper) half plane of more compliant material (i.e., ice). Here we derive a suitable value for $\zeta$ based on the material parameters listed in Table 1. We use subscript “bed” in order to denote the lower half plane (i.e., $y \leq 0$). Due to the problem symmetry about $x=0$, only one half of the domain, i.e., $x \in [0, \infty)$, is considered.

Integrating equation (1) for the upper half plane from $x$ to $\infty$, and imposing $\Delta w(x \to \infty, t)=0$, we obtain

$$\Delta w(x, t) = -\frac{2}{\pi E'} \int_x^{\infty} \left[ \int_{-\infty}^{\infty} \frac{\Delta p(s, t)}{s-x} \ ds \right] \ dx. \quad (A1)$$

A similar equation can be written for $\Delta w_{\text{bed}}(x, t)$ as well. Because the deformation of both half planes are subject to the same $\Delta p(x, t)$, we obtain the following relationship between $\Delta w$ and $\Delta w_{\text{bed}}$:

$$\Delta w_{\text{bed}}(x, t) = \frac{E'}{E_{\text{bed}}} \Delta w(x, t). \quad (A2)$$

Hence, the total elastostatic opening $w_\xi$ is given by

$$w_\xi(x, t) = \xi \Delta w(x, t). \quad (A3)$$

where

$$\xi = 1 + \frac{E'}{E_{\text{bed}}} \approx 1.104 \quad (A4)$$

for the choice of material parameters listed in Table 1. As expected, $\xi \to 1$ in case of rigid bedrock (i.e., $E_{\text{bed}} \gg E'$) and $\xi = 2$ in the situation with englacial conduit.

Note that the bimaterial approximation presented above is equivalent to the more complicated work of Tsai and Rice [2010], following Rice and Sih [1965], for a crack in a plane along the interface of dissimilar materials.
Appendix B: Parameter-Space Analysis and Optimized Mapping Parameters

For simulations with a given time step, how well the objective function (equation (33)) can be optimized (or, equivalently, how accurate the solutions are) potentially depends upon the choice of three parameters, namely, J, N, and l. Parameter J does not seem to affect the solutions as long as a sufficiently large number of quadrature points are used. We find that J = 30 is a reasonable choice for all of our calculations. Note that the quadrature points are also mapped with l according to equation (34) and therefore vary during simulations of flood propagation.

For a fixed choice of J, the magnitude of \( R^2 \) characterizes the optimization quality or solution accuracy. We evaluate the relative roles of \( N \) and \( l \) on the optimization quality for one example setting: flooding through the hydrological network with \( w_i = 1 \) cm after 2.5 min of drainage. We choose limits for parameter-space as follows: \( N \in [3, 6] \) and \( l \in [0.15, 0.45] \). A larger upper limit for \( N \) could have been considered, but we find it computationally expensive. For the chosen upper limit (i.e., \( N = 6 \)), the total number of terms in the series is 13 (see, e.g., equation (23)). The range for \( l \) is chosen after identifying through a separate experiment that the best optimization is achieved with \( l \approx 0.30 \).

We plot \( R^2 \) versus \( l \) in Figure B1a for different choices of \( N \). Clearly, optimizations are highly sensitive to both \( N \) and \( l \). For every \( N \), there exists a unique \( l \) that yields the best optimization. We term it the "optimal" mapping parameter. If \( l \) deviates farther from the optimal value, \( R^2 \) increases monotonically. Results also suggest that better optimizations are achieved, as expected, with more terms in the series. For larger \( N \), the range over which the given accuracy can be obtained is relatively wider. To achieve \( R^2 < 10^{-3} \), for example, \( l \) should be \( \approx 0.29 \) for \( N = 3 \), but any value in the range \([0.27, 0.31]\) for \( N = 6 \). In this sense, finding the optimal \( l \) is less critical for larger \( N \).

We also present a few examples of optimized solutions (Figure B1b). The LHS (equation (31)) and RHS (equation (32)) of the fluid flow equation are compared at corresponding Gauss-Chebyshev quadrature points. Solutions associated with the optimal mapping parameter (i.e., \( l = 0.29 \)) reveal better fits for larger \( N \) (notice a slight misfit for \( N = 3 \) near \( x = 1 \)). Similarly, the figure reveals poor fits for a case with nonoptimal mapping parameter (i.e., \( l = 0.35 \)) even though large \( N \) is considered.
In summary, we note that a large number of terms $N$ should be considered in the series if possible. We use $N=6$ for all of our calculations. The parameter $l$ is tuned until the optimal value is identified, so that the objective function is best optimized.

In the end, it might also be useful to briefly remark on the mapping parameters, identified to yield the optimized solutions that are presented in section 4. Figure B2 depicts how the optimal mapping parameter evolves in different scenarios. For all choices of $w_i$, the optimal $l$ evolves almost linearly as the floods propagate. Larger optimal $l$ is needed for greater $w_i$, as expected, in order to capture the relatively larger widths of the solutions (see Figures 5 and 7). Note that optimal $l$ and characteristic solution widths have similar orders of magnitude.

**Notation**

- $a_n$: unknown coefficients appearing in series.
- $D$: series constant chosen to satisfy $\Delta \hat{p}(0, \hat{t}) = 1$.
- $E$: Young's modulus of elasticity.
- $E'$: effective modulus of elasticity.
- $E'_{\text{bed}}$: effective modulus of elasticity for bed.
- $f_0$: Darcy-Weisbach friction factor coefficient.
- $g$: acceleration due to gravity.
- $G_n$: functional of $TB_n$ (equation (26)).
- $H$: height of the water column.
- $H_{\text{ice}}$: ice thickness.
- $\hat{H}_{\text{ice}}$: normalized ice thickness.
- $J$: number of Gauss-Chebyshev quadratures.
- $k$: Nikuradse roughness height.
- $K_i$: stress intensity factor.
- $K_{\text{IC}}$: fracture toughness.
- $l$: mapping parameter used in the series.
- $L$: half length of subglacial crack.
- $LHS$: LHS of the fluid flow equation.
- $N$: characterizes number of terms in series.
- $p$: water pressure.
- $\Delta \hat{p}$: excess pressure.
- $\Delta \hat{p}_{0}$: characteristic scale for $\Delta \hat{p}$.
- $\Delta \hat{p}$: nondimensional $\Delta \hat{p}$.
- $\Delta \hat{p}_n$: basis functions for $\Delta \hat{p}$.
- $\Delta \hat{p}^{\text{ss}}$: $\Delta \hat{p}$ for hydrofracture solutions.
- $q$: fluid flux.
- $q_0$: characteristic scale for $q$.
- $\hat{q}$: nondimensional $q$.
- $Q_0$: total flux injected at time $t$.
- $r$: radial distance away from the dislocation.
- $R^2$: squared 2-norm of the residual.
- $RHS$: RHS of the fluid flow equation.
- $t$: time.
- $t_0$: characteristic timescale.
- $\hat{t}$: nondimensional $t$.
- $\Delta \hat{t}$: nondimensional time step.
- $T_n$: Chebyshev polynomials of the first kind.
- $TB_n$: rational Chebyshev functions.
- $v$: depth-averaged fluid velocity.
- $v_0$: characteristic scale for $v$.
- $\hat{v}$: nondimensional $v$.
- $w$: total ice/bed opening.
- $w_c$: total crack opening.
\( w_c \) total elastostatic opening.

\( w_i \) initial opening.

\( w_0 \) characteristic scale for opening.

\( \hat{w} \) nondimensional \( w \).

\( \hat{w}_c \) nondimensional \( w_c \).

\( \hat{w}_{\text{n}} \) normalized \( w \) for hydrofracture solutions.

\( \Delta w \) vertical displacement of half plane.

\( \Delta w_{\text{bed}} \) vertical displacement of lower half plane.

\( \Delta \hat{w}_n \) basis functions for \( \Delta \hat{w} \).

\( x \) horizontal position, defined as in Figure 1.

\( x_0 \) characteristic length scale.

\( \hat{x} \) nondimensional \( x \).

\( \chi \) function of \( x \) and \( j \) (equation (27)).

\( \hat{\chi} \) \( \chi \) for hydrofracture solutions.

\( y \) vertical position, defined as in Figure 1.

\( z \) arbitrary point on the half plane or a complex variable.

\( \chi \) a signum function (equation (5)).

\( \varepsilon \) small nondimensional number.

\( \nu \) Poisson’s ratio.

\( \rho \) water density.

\( \rho_{\text{ice}} \) ice density.

\( \sigma_0 \) ice overburden pressure.

\( \xi \) bimaterial correction factor.

\( \zeta \) \( \zeta \) for hydrofracture solutions.

References


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