Effects of solid-solid phase changes on impact mechanics

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Abstract: This paper describes an idealized problem that models the response of an elastic solid undergoing a shear-induced phase transformation due to dynamic loading.

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1. Introduction

Elastic materials whose strain-energy potentials are non-convex functions of strain can be used to model the macroscopic effects of solid-solid phase changes (Abeyaratne and Knowles 1990, 1991). In the presence of such potentials, the boundary initial-value problems of finite elasticity fail to possess unique solutions, despite enforcement of the second law of thermodynamics. It has been shown that uniqueness is restored if a nucleation criterion governing the initiation of the phase transition and a kinetic relation governing its evolution are added to the field equations and jump conditions of elastodynamics (Abeyaratne and Knowles 1991).

We describe a model problem involving dynamic loading of an infinite elastic medium subject to a concentrated shear force whose magnitude increases linearly in time at a given rate. Because the problem involves a concentrated load generating singularly large strains and stresses, nucleation always occurs, but imposition of a kinetic relation remains necessary.

We compare the distribution of energy densities in the medium in the presence of a phase change with the corresponding distribution in a material incapable of phase change.

2. The problem

Consider an infinite elastic medium, initially undeformed and at rest in a configuration that is taken as reference. Let \( r, \theta, z \) be cylindrical coordinates in this configuration (Fig. 1). At time \( t = 0 \), an axially directed shear traction is applied as a line load along the \( z \) axis, its magnitude at time \( t \) being
At, where the loading rate $\lambda$ is given. The deformation is taken to be finite anti-plane shear. The only nonvanishing component of displacement is the axial component, $u_r = u(r, t)$, and the corresponding shear stress of interest is $\sigma_{rz} = \sigma_{rz} = \sigma(r, t)$. The material we consider is an incompressible neo-Hookean solid (a material which can sustain finite anti-plane shear) which possesses the shear stress-shear strain relationship of Fig. 2. This curve corresponds to an elastic material whose strain energy potential is a non-convex function of strain. The strains and stresses engendered by the concentrated loading at hand will be singular at $r = 0$, and hence will always be sufficient to nucleate a phase change from phase 1 to phase 3 there, the unstable phase being absent (Abeyaratne and Knowles 1991). The interface between the low-strain phase and the high-strain phase, the phase boundary, is located at $r = st$ at time $t$, where $s$ is the phase-boundary velocity, assumed constant. We assume the phase boundary propagates subsonically, so that $0 < s < c_0$. The expanding phase boundary is preceded by a shear wave; the expected wave pattern at time $t$ is as shown in Fig. 1.

![Figure 1](image1.png)

Figure 1. The problem: a shear traction is applied along the $x$ axis (out of page) and ramped from 0 at rate $\lambda$. A shear wave propagates radially, possibly followed by a phase boundary.

![Figure 2](image2.png)

Figure 2. Stress-strain curve of the material.

3. The boundary initial-value problem

Away from strain discontinuities, the classical field equations of finite elastodynamics (Abeyaratne and Knowles 1990) reduce to:

$$c_0^2 \nabla^2 u(r, t) - \ddot{u}(r, t) = 0, \quad \text{for } 0 < r < st,$$

where $\ddot{u}(r, t)$ is the acceleration of the material at $r$. If $u(r, t)$ satisfies the boundary conditions $u(0, t) = 0$ and $\sigma_{rz}(r, t) = 0$ for $r = st$, then $u(r, t)$ is the solution.
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\[ c_1^2 \nabla^2 u(r, t) - \dot{u}(r, t) = 0, \quad \text{for } st < r < \infty, \]

where \( c_1 = \sqrt{\mu_1 / \rho} \) and \( c_2 = \sqrt{\mu_2 / \rho} \) are shear wave speeds in the low- and high-strain phases, respectively, and \( \rho \) the referential mass density. The initial and boundary conditions are:

\[ u(r, 0^+) = 0, \quad \dot{u}(r, 0^+) = 0, \]

\[ \lim_{\tau \to 0} 2\pi \tau \sigma(\tau, t) = \lambda'. \]

Across the strain discontinuities at the phase boundary, \( r = st \), and at the wave front, \( r = c_1 t \), the jump conditions of elastodynamics specialize to

\[ u(c_1 t^+, t) - u(c_1 t^-, t) = 0, \quad u(\dot{c}t^+, t) = u(\dot{c}t^-, t), \]

\[ c_1 \dot{c}_1 \theta(c_1 t^+, \dot{c}t^-) - c_1 \dot{c}_1 \theta(c_1 t^-, \dot{c}t^+) + \rho \dot{\theta}[u(\dot{c}t^+, t) - u(\dot{c}t^-, t)] = 0. \]

4. The solution for constant \( \dot{c} \)

As posed, the problem has a similarity solution, in which

\[ u(r, t) = \begin{cases} c_1 \dot{c}_1 \theta(r / (c_1 t^+), \dot{c}t^-), & \text{for } 0 < r < \dot{c}t, \\ c_1 \dot{c}_1 \theta(r / (c_1 t^-), \dot{c}t^+), & \text{for } \dot{c}t < r < c_1 t, \\ 0, & \text{for } c_1 t < r, \end{cases} \]

where \( \dot{c}_1 \) and \( \dot{c}_2 \) are fully and explicitly determined functions of \( r / (c_1 t) \) and \( \dot{c} \), the formulas for which are omitted here for brevity; a more detailed account of this problem will appear elsewhere. Thus, Eq. (7) provides a one-parameter family (parameter \( \dot{c} \)) of solutions of our boundary initial-value problem. To determine \( \dot{c} \), a kinetic relation must be invoked, as sketched below.

The solution (Eq. 7) must be such that the associated strains \( \gamma \) outside and inside the moving phase boundary lie in the intervals corresponding respectively to phases 1 and 3 (Fig. 2). This requirement ultimately leads to the restriction that the point \( (\dot{c}, \lambda) \) lie in the shaded region of Fig. 3.

Not every member of the family of solutions (Eq. 7) conforming to Fig. 3 satisfies the second law of thermodynamics. When this restriction is imposed in the form described in Abeyaratne and Knowles (1990), it leads to the requirement that \( (\dot{c}, \lambda) \) lie in the shaded region shown in Fig. 4. Thus, prior to the imposition of a kinetic relation, only those members of the one-parameter family of solutions (Eq. 7) corresponding to the shaded region in Fig. 4 satisfy the field equations, the jump conditions at the phase boundary and at the shear wave, the phase-segregation conditions, and the entropy inequality.

The curves forming the boundaries of the shaded regions in Figs. 3 and 4 can be fully and explicitly determined from the similarity solution (Eq. 7).
5. The kinetic relation

As shown in Abeyaratne and Knowles (1990, 1991), an appropriate way to describe the kinetics of the phase transition in the present setting is to require that the driving force at the phase boundary be a specified function, determined by the material, of the phase boundary velocity $\dot{s}$. By driving force, we mean the generalization to dynamics of the jump across the phase boundary of the Gibbs free energy of quasi-statics; see the discussions in Abeyaratne and Knowles (1990, 1991). When this notion is properly implemented in the present problem, the set of points $(\dot{s}, \lambda)$ admitted by the kinetic relation consists of a curve within the shaded region in Fig. 4. This curve allows one to uniquely determine a phase-boundary velocity $\dot{s}$ for each given load rate $\lambda$ in a certain range, and hence to single out a unique member of the one-parameter family of solutions (Eq. 7) to the underlying boundary-initial-value problem.

6. Energy distribution

The stored elastic energy $W$ per unit reference volume for our elastic material at a strain $\gamma$ is given by the area under the stress-strain curve between 0 and $\gamma$ (Fig. 2). The total energy per unit volume is given by $E = W + (1/2)\rho u^2$; here, $E$ depends on position $r$ and time $t$. For each solution among the one-parameter family (Eq. 7), $E$ can be determined as a function of $r, t,$ and the phase-boundary velocity $\dot{s}$. The value of $\dot{s}$ must be determined by the kinetic relation. Here, however, we assume $\dot{s}$ to be given, and in Fig. 5 we plot $E$ versus $r$ for a given $t$ and for a given $\dot{s}$ for one of the phase-transition solutions (Eq. 7).

For comparison, we consider a second elastic material in which the stress-strain relation replacing that of Fig. 2 is $\sigma = \mu_1 \gamma$ for all strains $\gamma$. Thus our

![Figure 3. Shaded region indicates valid solutions after enforcing phase-segregation requirements.](image)

![Figure 4. The second law eliminates more solutions.](image)
second material is capable of existing in a single phase only, and its properties in this phase are identical with those of our two-phase material in its low-strain phase. The solution of the present boundary initial-value problem for this new material is obtained by standard methods, and from this solution, one may compute the distribution of total energy density as a function of \( r \) for each \( r \) for purposes of comparison with the energy density distribution for the two-phase material described above (Fig. 5).

Motivated by applications to ceramic armor, we consider the total energy density in the untransformed material ahead of the phase boundary and compare it to the energy density of the same particles in the single-phase material. \( E_1 \) and \( E_2 \) are the respective one-phase and two-phase total energy densities in this region; we plot the ratio \( E_2/E_1 \) in Fig. 6. This quantity is a function of only the ratio of shear wave speeds of the two phases, \( c_3/c_1 \), and of the ratio of the phase-boundary velocity to the shear wave speed in the original phase, \( v/c_1 \).

Figure 5. Total energy density in the trilinear material and in a single-phase material. The single-phase material has the properties of phase 1.

Figure 6. Ratio of total energy density ahead of the phase boundary in trilinear material to that of same particles in single-phase material.

7. Concluding remarks

A phase transformation in the present idealized model decreases the total energy to which the untransformed material is exposed. This "shielding" effect is enhanced with an increasing shear modulus in the original phase and with a decreasing shear modulus in the new phase. A faster phase-boundary velocity also increases the shielding effect. In this model, the phase-boundary velocity cannot exceed the shear-wave speed of the new phase.
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References