Observed Heat Flows

These are only available when the surface has been instrumented (Earth, Moon) or the heatflow is so high that it can be detected by excess IR output (giant planets and Io). The Sun is included for comparison purposes as is reradiation of sunlight at Earth (a blackbody at 240K).

<table>
<thead>
<tr>
<th>Body</th>
<th>Heat flux (erg/cm²·sec)</th>
<th>Luminosity (erg/sec)</th>
<th>Luminosity per unit mass (erg/g·sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>$6.3 \times 10^{10}$</td>
<td>$4 \times 10^{33}$</td>
<td>2</td>
</tr>
<tr>
<td>240K Earth</td>
<td>$1.9 \times 10^{5}$</td>
<td>$1 \times 10^{24}$</td>
<td>$1.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>(Black body)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>80</td>
<td>$4.3 \times 10^{20}$</td>
<td>$7 \times 10^{8}$</td>
</tr>
<tr>
<td>Moon</td>
<td>~17</td>
<td>~7 $\times 10^{18}$</td>
<td>~9 $\times 10^{8}$</td>
</tr>
<tr>
<td>Io</td>
<td>~2500</td>
<td>~1 $\times 10^{21}$</td>
<td>~1 $\times 10^{-5}$</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5400</td>
<td>$3 \times 10^{24}$</td>
<td>$1.7 \times 10^{-6}$</td>
</tr>
<tr>
<td>Saturn</td>
<td>2000</td>
<td>$8 \times 10^{23}$</td>
<td>$1.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>Uranus</td>
<td>&lt;180</td>
<td>&lt;1.5 $\times 10^{22}$</td>
<td>&lt;1.7 $\times 10^{-7}$</td>
</tr>
<tr>
<td>Neptune</td>
<td>285</td>
<td>$2.2 \times 10^{22}$</td>
<td>$2.2 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Expected Heat Flows

Radioactivity

By mass, the cosmic abundances of K, U, and Th are in the ratios 60,000:4:1. For those ratios (close to those actually observed in CI carbonaceous chondrites) the expected present day heat production of carbonaceous chondritic material is around 6 or $7 \times 10^{-8}$ erg/g·sec. This comes roughly one half from $^{40}$K, and one quarter each from $^{238}$U and $^{232}$Th. [Warning: Stacey’s textbook provides values for K in CI that are too low by a factor of up to 6; for more reliable numbers look at the various tables in Lodders and Fegley]. The CI value for heat production looks roughly Earthlike but that is a fallacy because Earth’s mantle is depleted in K by
about an order of magnitude relative to CI. There is no reason to suppose that Earth’s core contains large amounts of potassium; the depletion is more reasonably attributable to volatility. (Other elements of similar volatility to K are also depleted in Earth and there is no expectation that they would go into the core). Consequently, Earth’s energy output is believed to be roughly a factor of two larger than that due to radioactivity alone. Moon’s heat flow is too poorly determined to yield to any useful analysis. It may be anomalously high in the places measured (or just very poorly measured).

Note two important things about radioactive heating: First, that it will scale as the mass of the planet (which means that the surface heat flux will scale roughly as mass over area or, equivalently, radius.) So Mars would have a factor of two lower heat flux than Earth (all else being equal, which it isn’t!) Second, radioactive heat decays with time so the heat flow should be much larger in the early history of the planet. For Earth, radioactive heating should be about four times larger in early history (with $^{40}$K and $^{235}$U playing major roles).

Clearly, none of the heat flows measured in the outer solar system can be attributed to radioactivity, especially when you consider that the mass of these bodies (except Io) is dominated by material (H, He, C, N, O) that contains no radioactive elements.

**Secular Cooling**

Suppose a planet has cooled by an amount $\Delta T$ over 4.5 billion years. If it has a mean specific heat $C_v$, then the expected luminosity per unit mass, $L/M$, is obviously $C_v \Delta T/(4.5 \times 10^9 \text{yr})$ which is:

$$L/M = (7 \times 10^{-8} \text{ erg/g/sec}).(C_v/1 \times 10^7 \text{cgs}).(\Delta T/10^3 K)$$

For Earth, this will be significant, or even dominant for cooling rates of ~200K/billion years. For Jupiter and Saturn where the specific heat is ~2 $\times 10^8$ cgs (because of the low molecular weight), the observed luminosity can be explained for total cooling of a few thousand degrees (though, as we shall see, the cooling may be fast early on). Neptune could also be explained with ~1000-2000K cooling and an ice-dominated specific heat ($C_v \sim 2 \times 10^7$).
Differentiation

If some fraction $x$ of a planet settled to the bottom and was ~twice as dense as mean density then you might expect the total energy release to be $\sim xGM^2/R$. Averaged over $4.5 \times 10^9$ yrs, one then has

$$L/M \sim 1 \times 10^{-4} x \text{ for Jupiter}$$
and $\sim 3 \times 10^{-5} x \text{ for Saturn},$

so this could easily be important. For Earth, one gets $\sim 4 \times 10^{-6} x$, which could be important but there is no current differentiation that would yield significant $x$ (core formation was early in Earth’s history).

Tides

This is special to Io and Europa (and maybe for part of Ganymede’s history) but has to be evaluated on an individual basis because it is sensitive to orbital parameters as well as material properties.

The Conductive (or Radiative) State

We can use our knowledge of plausible planetary materials to pose the following question: What would the thermal state be inside a planet, were the heat to be carried by microscopic (i.e. conductive or radiative) processes alone? (This is a separate but related question to the one we discussed earlier of whether there is sufficient time to set up a steady state that depends on microscopic transport.)

A. Terrestrial Planets

The typical thermal conductivity of mantle rocks and crust is around $3 \times 10^5$ erg/cm.s.K. This means that for a heat flow $F$, the increase of temperature with depth $z$ is given by:

$$\frac{dT}{dz} (K/km) = \frac{F(\text{erg/}cm^2.s)}{3}$$

which implies a temperature gradient of 20K/km or even more with depth for Earth. Even on the Moon, one gets 5 or so K/km. On Earth, this
temperature gradient leads to a temperature at depth D (in km) of around 300+20D, which exceeds the melting point (about 1600K) at about 65 km. *But we know that earth is not pervasively molten at this (or any mantle) depth!* We know this from seismology. You can reduce but not eliminate the problem by arguing that the heat sources are concentrated near the surface (so that the temperature profile is not linear). However, this cannot solve the problem because we know the radioactivity of mantle rocks (from xenoliths) and it provides a large fraction of the heat flow. So the conclusion for Earth (and by plausible extension other large terrestrial bodies) is that the heat cannot get out entirely by conduction. It is less clear for the Moon.

In Io, the implied temperature gradient is even greater, but clearly most of the heat gets out volcanically (i.e. by magmatic eruptions and lava flows). For example, you can explain 2500 erg/cm².sec, by the delivery of ~3 x10¹⁰ erg/cm³ (the expected heat content of magma) at a mean “velocity” (i.e., resurfacing rate) of ~2cm/yr (= 7 x 10⁻⁸ cm/sec).

**B. Giant Planets**

The situation here is far less straightforward. First, consider the deep atmosphere. These bodies are dominated by molecular hydrogen, which also usually provides the dominant opacity source through pressure-induced absorption. The property of pressure-induced absorption is that the opacity is proportional to pressure. Now the heat flow carried by radiation can be written in the form

\[ F_{\text{rad}} \approx d(\sigma T^4)/d\tau \]

where the increment in optical depth is \( d\tau = -\rho \kappa dr \) and \( \kappa \) is the opacity. (This equation should be intuitively obvious to order of magnitude since the photons go one optical depth before being absorbed). Optical depth is dimensionless and here defined to decrease as one goes outwards, so an optical depth of one corresponds to the place from which photons escape to space. This predicts that if radiative transport dominates, then \( T \sim T_e \tau^{1/4} \) where \( T_e \) is the temperature at optical depth unity (measured from the outside inwards).

From hydrostatic equilibrium, \( dp/dr = -\rho g \) and the definition of optical depth, we have \( dp/d\tau = g/\kappa \). But in molecular hydrogen, the opacity is roughly proportional to pressure and so integrating, we have \( p \sim \tau^{1/2} \) and
constancy of $F_{\text{rad}}$ then implies $T \sim p^{1/2}$. This is a stronger dependence than the increase of temperature along an adiabat. Radiative transport in molecular hydrogen would therefore require that the entropy decrease with height. As we see below, this leads to a convective instability. [Note: This argument may break down around $T \sim 1000\,\text{K}$ where molecular hydrogen becomes somewhat transparent. In other words, the opacity does not just depend on pressure it also depends on the wavelengths of the thermal photons involved, or equivalently on temperature. Tristan Guillot has studied this in detail.]

Deep within the planet, the highest thermal conductivity you can find is that attributable to metallic hydrogen. This is certainly less than $10^9\,\text{erg/cm.s.K}$. Even for this upper bound, the observed heat flow would lead to a temperature gradient of $0.3\,\text{K/km}$, implying a temperature increase of around $20,000\,\text{K}$ for a radial range of $60,000\,\text{km}$. As we shall see, this is unstable. The situation in dense molecular hydrogen is much worse... a thermal conductivity of perhaps $10^7\,\text{erg/cm.s.K}$ and a conductive gradient of around $30\,\text{K/km}$, implying ridiculous temperature increase with depth.

In all planets, the conductive profiles are usually convectively unstable. To understand this, we need some fluid dynamical background.

**A Fluid Dynamical Preliminary**

In all fluid dynamics, we are principally interested in understanding the solutions of the well-known equation

$$F = ma$$

where $F$ is the force, $m$ is the mass and $a$ is the acceleration. Of course, we are usually interested in continua rather than discrete masses, so we more commonly write this equation in the form

$$a = f/\rho$$

where $f$ is the force per unit volume and $\rho$ is the mass density (mass per unit volume). It is common (but not essential) to choose a fixed coordinate grid and talk about the acceleration of the medium at a grid point. As time progresses, the material flows pass any given grid point. However, $a$ is by definition the acceleration of an element of material that happens to be at a
given grid point at a given time. Accordingly, it is not merely the time rate of change of velocity at that point, but must include the usual advection term:

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \]

where \( \frac{d.../dt \) is called the \textit{convective derivative} and is the derivative in the comoving frame of the fluid element.

The greatest richness of phenomenology lies in the contributions to \( \mathbf{f} \), the force acting on a unit volume of the medium. These contributions are of two kinds: \textit{body} forces and \textit{surface} forces. As the words imply, a body force is one that acts throughout the elementary volume, while a surface force is the sum of external forces acting on the element due to adjacent elements. The archetypal body force is gravity; the archetypal surface force arises from a pressure or stress \textit{gradient}.

Let's write the total force in the form

\[ \mathbf{f} = \mathbf{f}_b + \mathbf{f}_s \]

representing body and surface forces respectively. Gravity is by far the largest body force in situations of interest to us, though as we shall see nearly all of it is hydrostatically balanced and thus not of dynamical interest. So we have

\[ \mathbf{f}_b = \rho \mathbf{g} \]

The surface force has a form that depends on the \textit{constitutive law} of the medium: is it elastic, or plastic or viscous or what? This is a major issue, but for now we will assume a simple linear viscous model:

\[ \sigma_{ij} = -p \delta_{ij} + \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

\[ \mathbf{f}_s = \text{div}(\sigma_{ij}) \]

where \( \sigma_{ij} \) is the stress tensor, \( p \) is the pressure, \( \delta_{ij} \) is the Kroenecker delta (0 if \( i \neq j \); 1 if \( i = j \)), and \( \eta \) is the \textit{dynamic} shear viscosity (Poise in cgs units)
and Pa.s in SI units). Actually, this is not the most general linear viscous model, since it includes only shear viscosity.

If there is no fluid flow, then \( \mathbf{f}_b + \mathbf{f}_s = 0 \); \( \mathbf{f}_b = \rho g \) and \( \mathbf{f}_s = -\nabla p \), so the solution is

\[
\nabla p = \rho g
\]

which is, of course, the equation of hydrostatic equilibrium. Now here's an interesting thing: This equation does not have a solution for all possible density distributions. To see this, take the curl

\[
0 = \nabla \rho \times \mathbf{g}
\]

where I have made use of the fact that \( \nabla \times \mathbf{g} = 0 \), (because \( \mathbf{g} = \nabla \phi \), where \( \phi \) is the gravitational potential). This equation will not be satisfied if there is any variation of the density on an equipotential surface. As we shall see later, it is precisely such a variation that drives convection and many other dynamical flows.

The equation of motion is never enough to fully characterize the flow. Several other ingredients are needed, one of which is the equation of continuity. It expresses a very simple statement physically: The mass of material in a unit volume changes according to the net flux of mass into that volume:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})
\]

It is often adequate to adopt the incompressible flow assumption, in which

\[
\nabla \cdot \mathbf{u} = 0
\]

If we assume incompressibility for the viscous response, then the equation of motion becomes

\[
\frac{d\mathbf{u}}{dt} = -\nabla p/\rho + \mathbf{g} + \nu \nabla^2 \mathbf{u}
\]

where \( \nu \equiv \frac{\eta}{\rho} \) is the kinematic viscosity and has units of a diffusivity (e.g. cm\(^2\)/sec.) It has been assumed here that \( \eta \) is independent of position.
If we take the curl of this equation and consider only small velocities and small perturbations away from the mean density then we get:

$$\frac{\partial (\nabla \tilde{u})}{\partial t} = \frac{\nabla (\delta \rho) x \tilde{g}}{\rho_0} + v \nabla^2 (\nabla \tilde{u})$$

where the non-linear term is dropped as being small and only density perturbations arising from the flow contribute to vorticity generation.

**Rayleigh-Taylor (or Interchange) Instabilities**

Consider first an incompressible fluid in which there is a density gradient vertically (arising from temperature differences or compositional differences or whatever). The density is accordingly

$$\rho = \rho_0 (1 + \beta z)$$

where $z$ is measured upwards relative to some arbitrary reference level. We suppose that an element of material which is displaced from rest will accordingly see a density difference relative to the surroundings.

In other words, the hydrostatic state need no longer be satisfied. Let all variables behave like $\exp(ik\cdot r + \sigma t)$ and let the density disturbance be $\delta \rho$. Then by continuity,

$$\sigma \delta \rho = -\ddot{u} \cdot \nabla \rho = -u_z \rho_0 \beta$$

The equation of motion now becomes

$$\sigma (i \kappa \tilde{x} \tilde{u}) = -i \kappa \tilde{x} \tilde{g} (u_z \beta / \sigma) - v \kappa^2 (i \kappa \tilde{x} \tilde{u})$$
Taking yet another curl (i.e. \( i\mathbf{k} \)) and remembering that \( \mathbf{k}.\mathbf{u}=0 \) (incompressibility), we get:

\[
(\sigma + \nu k^2) \mathbf{k}^2 \mathbf{u} = -[k^2 \mathbf{g} - k \mathbf{k}.\mathbf{g}](u_\beta / \sigma)
\]

Finally, we take the z-component of this (remembering that the vector \( \mathbf{g} \) is in the negative z-direction) and we get:

\[
\sigma(\sigma + \nu k^2) = g\beta(k_x^2 + k_y^2)/k^2
\]

which has growing solutions for all choices of the wavevector, provided \( \beta>0 \).

**A. The Inviscid Limit**

In this limit, \( \sigma \gg \nu k^2 \), the viscous term is ignored and we get \( \sigma = \sqrt{g\beta} \). This has a simple mechanical explanation. If we have an element that has been displaced a distance \( z \) vertically then it feels a force which accelerates it at \( g\beta z \), so the equation of motion becomes \( \frac{d^2z}{dt^2} = g\beta z \), which has a solution like \( \exp(\sigma t) \) with \( \sigma = (g\beta)^{1/2} \). As an example, a 10% variation in density over 1000km (which means \( \beta=10^{-9} \text{ cm}^{-1} \)) gives an e-folding time for the instability of \( 10^3 \) seconds for Earth... very fast! It follows that unstable density gradients in low viscosity fluids on planets (atmosphere, ocean core) relax to small values. Of course, negative gradients cause oscillations (imaginary values for \( \sigma \)).

**B. The Viscous Limit**

In this limit, \( \sigma \ll \nu k^2 \), and the solution becomes

\[
\sigma \approx \frac{g\beta}{\nu k^2}
\]

This is the Stokes flow limit in which the forces are balanced and the inertial term \((\mathbf{u}/\mathbf{t})\) is negligible, as one can confirm by going back to the original equation of motion. As an example, for Earth \((g \sim 10^3 \text{ cm/s}^2, \ \beta \sim 10^{-9} \text{ cm}^{-1} \) which is a 10% change in 1000km, \( \nu \sim 10^{21} \text{ cm/s}^2 \), and \( k \sim 6 \times 10^8 \text{ cm}^{-1} \) which is a wavelength of 1000km) one gets a characteristic timescale \((\sigma^{-1})\) of \( \sim 10^5 \) yrs. Note that in this case (unlike the inviscid case) negative values of the density gradient cause decay (negative \( \sigma \)) rather than oscillation.
(imaginary sigma). This corresponds to processes such as post-glacial rebound.

Relationship to Adiabats and the Convective Instability

Now consider a fluid that has a temperature gradient that is steeper than the adiabat. Upon displacing an element and assuming no heat transfer horizontally, it is clear that positive $\beta$ as above corresponds to a superadiabatic temperature profile. This is the convective instability, which is a special case of the Rayleigh-Taylor instability (special because in general one has to worry about the loss of the buoyancy due to conduction).