The Gravity Field

External to a planet (in a region where there is no or negligible mass), the gravitational potential \( V \) satisfies Laplace’s equation

\[
\nabla^2 V = 0
\]

It makes sense to use a planet-centered spherical coordinate system, and in that case the general solution to Laplace’s equation can be written in the form:

\[
V = \frac{1}{a} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{a}{r} \right)^{l+1} \left( C_{lm} \cos m\phi + S_{lm} \sin m\phi \right) P_l^m(\cos \theta)
\]

[Warning: There are various ways of writing this and defining the coefficients in this expansion. Some people use a minus sign in front of the whole thing. Some people use complex coefficients, i.e. \( \exp(im\phi) \) instead of sines and cosines. You can always check your sign convention by making sure that the gravitational acceleration is pointing in the right direction, but to get the rest right you have to find out the author’s normalization convention, etc.] The solution assumes no external sources of mass (i.e. all terms decay as \( r \) goes to infinity). The equatorial radius of the planet is \( a \). The \( P \)'s are associated Legendre functions, and along with the sines and cosines of longitude they define spherical harmonics usually written \( Y_{lm} \), so the \( C \)'s and \( S \)'s are called spherical harmonic coefficients.

Obviously, \( C_{00} \) is nothing other than \( GM \). (By the way, \( GM \) can be measured to twelve figure accuracy for Earth, but that doesn’t mean we know \( M \) that well! \( G \) is the least well known fundamental constant and very hard to measure). Precise tracking of an orbiting spacecraft can give you these coefficients. As in any mathematical representation, you always truncate the representation and thus fail to characterize very high harmonics. This is a potential problem even in quite low planetary orbits.
In the special case where the planet is a rotating hydrostatic fluid, symmetry arguments alone dictate that the potential is axisymmetric (only $m=0$ terms allowed) if we choose our polar axis to be coincident with the rotation axis. Moreover, there is no physical distinction between north and south hemispheres, so odd $\ell$ values are excluded. (This assumes we’ve chosen the origin of coordinates to be the center of mass). We can then write the potential in the form:

$$ V = \frac{GM}{r} \left[ 1 - \sum_{\ell=1}^{\infty} J_{2\ell} \left( \frac{a}{r} \right)^{2\ell} P_{2\ell}(\cos\theta) \right] $$

In this simple case, the $P$’s are now the simple Legendre polynomials, and the $J$’s are called gravitational moments. In rapidly rotating planets, $J_2$ is generally far larger than any of the other harmonics (except of course $C_{00}$).

The fundamental definition of the gravitational potential is of course

$$ V(\vec{r}) = G \int_{all\ space} \frac{\rho(\vec{r}')d^3r'}{|\vec{r} - \vec{r}'|} $$

obtained by adding up the contributions of all masses and appealing to the superposition principle (the linearity of Newtonian gravity). Outside the planet, we can appeal to the fundamental theorem (also known in mathematics as the generating function):

$$ \frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_{\ell}(\cos\gamma) $$

where $\gamma$ is the angle between vectors $\vec{r}$ and $\vec{r}'$, and $\vec{r}$ is outside the planet ($\vec{r}'$ is inside the planet). Moreover,

$$ P_{\ell}(\cos\gamma) = \sum_{m=-\ell}^{\ell} (-1)^{m} \frac{(\ell - m)!}{(l + m)!} P_{\ell}^m(\theta) P_{\ell}^m(\theta') \cos[m(\phi - \phi')] \cdot $$

But we only need to keep track of $m=0$, for evaluating $J_{2\ell}$:

$$ P_{\ell}(\cos\gamma) = P_{\ell}(\cos\theta) P_{\ell}(\cos\theta') + m \neq 0 \text{ terms} $$

so it follows immediately that
\[ J_{2l} = -\frac{1}{Ma^2} \int r^{2l} P_{2l}(\cos \theta') \rho(r') d^3r' \]

One can see already why the J’s are called gravitational moments since they require higher order averages of the internal density distribution.

Recalling that \( P_2(\cos \theta) = (3\cos^2 \theta - 1)/2 \), one can see immediately that

\[ Ma^2 J_2 = -\int \left[ \frac{3}{2} z^2 - \frac{1}{2} (x^2 + y^2 + z^2) \right] \rho(r) d^3r \]

where \( z \) is along the rotation axis and \( x, y \) are in the equatorial plane. Now if we define the axial moment of inertia as \( C \) and the other two principal moments of inertia as \( A \) and \( B \) then we have:

\[ C \equiv \int (x^2 + y^2) \rho(r) d^3r \]
\[ A \equiv \int (z^2 + y^2) \rho(r) d^3r; \quad B \equiv \int (z^2 + x^2) \rho(r) d^3r \]

and it is easy to see that

\[ Ma^2 J_2 = C - \frac{1}{2} [A + B] \]

In the special (and highly relevant case) where \( A = B \) (i.e. the planet is a body of rotation rather than triaxial) we have

\[ Ma^2 J_2 = C - A \]

So this moment is related to the difference between axial and equatorial moments of inertia. In a similar manner you can show that \( C_{22} = -(B-A)/Ma^2 \).

But you cannot get the actual values of \( A, B \) and \( C \) from the gravity field; there is insufficient information. You can only get their differences (e.g. \( C-A \)).

The Response of a Planet to its Own Rotation

We saw that \( J_2 \) can be related to the difference in the moments of inertia about equatorial and polar axes. Together with the measurement of precession rates of a planet [which gives \( (C-A)/C \)], one can then solve for
the individual moments of inertia. This is how we know Earth, Mars and lunar moments of inertia. (Determination of the Mars’ moment of inertia using Mars Pathfinder tracking was one of the major accomplishments of that mission). But we ought to be able to figure out something more from $J_2$ alone because it’s value depends on how the planet responds to its own rotation, and this response depends on its density distribution. It should be obvious, for example, (just by looking at the definition as an integral over the interior weighted by radius squared) that $J_2$ will be small for a body that is centrally concentrated.

Although this is intuitively reasonable, the appropriate theory is quite nasty, and rather little insight emerges from wallowing in the nastiness. (The full theory involves integrodifferential equations that must be solved on a computer and even then converge slowly). I will only give a feeling for the theory. This will only work if $J_2$ is dominated by hydrostatic effects. The theory explicitly assumes hydrostaticity.

**The Constant Density Limit (Maclaurin Spheroid)**

Consider, first, the constant density body. The external potential is obviously completely determined by the shape of the free surface. Approximate the free surface of this body by the lowest order non-spherical shape permitted, i.e.

$$r_s = r_0 (1 + \varepsilon P_2)$$

where $r_0$ is some mean radius and epsilon is a dimensionless constant. Inserting in the fundamental equation for the gravitational potential and using the generating function (last chapter), one immediately finds that the only part that depends on $P_2$ is of the form

$$V_2 = \frac{GP}{r^3} \int_{-1}^{1} P_2 (\cos \theta') d(\cos \theta') \int_0^{\rho_0 (1+\varepsilon P_2)} x^2.2\pi x^2.\rho_0 .dx$$

Since the P’s are orthogonal to each other (and remember that $P_0$ is a constant), the only part of the integral over $x$ that contributes is the part proportional to $P_2$. Therefore:
\[ V_2 = \frac{GP_2}{r^3} \int_{-1}^{1} P_2^2(\cos \theta')d(\cos \theta') \cdot 2\pi \rho_0 a^5 \]

\[ = \frac{3G}{5r^3} Ma^2 \cdot \varepsilon \]

[where we’ve used the fact that the normalization integral for Legendre functions is \(2/(2l+1)\).] We must compare with the expression that defines \(J_2\) in terms of the external expansion of the field:

\[ V_2 = -\frac{GMa^2}{r^3} J_2 P_2 \]

\[ \Rightarrow J_2 = -\frac{3}{5} \varepsilon \]

Consider, now, the gravitational potential evaluated at the actual surface. We must of course include the effect of rotation (the “centrifugal” effect). Recall that the acceleration is \(\omega^2 s\) where \(s\) is the distance from the rotation axis. The potential that yields this acceleration is (by integration) obviously

\[ \frac{1}{2}\omega^2 s^2 \equiv \frac{1}{2}\omega^2 r^2 \sin^2 \theta \equiv \frac{1}{3}\omega^2 r^2 [1 - P_2] \]

Now the total potential must be constant at the surface. This is where the assumption of hydrostaticity enters. To lowest non-vanishing order in \(P_2\) this implies that:

\[ \frac{GM}{r_0(1 + \varepsilon P_2)} - \frac{GM}{a} J_2 P_2 + \frac{1}{3} \omega^2 a^2 [1 - P_2] \]

must have no dependence on \(P_2\). [Note: We can ignore the differences between \(r, a, \) etc. in terms that are already of order \(P_2\). Also, \(\varepsilon\ll1\) so \(1/(1+\varepsilon P_2) \approx 1 - \varepsilon P_2\) to an excellent approximation]. In other words,

\[ \left\{ \frac{GM}{a} [-\varepsilon - J_2] - \frac{1}{3} \omega^2 a^2 \right\} P_2 \equiv 0 \]

\[ \Rightarrow -\varepsilon - J_2 - \frac{q}{3} \equiv 0 \]

where \(q\) is a dimensionless measure of planetary rotation:

\[ q \equiv \frac{\omega^2 a^3}{GM} \]

But we already have \(J_2 = -3\varepsilon/5\). Substituting, we get
This is what we mean by the response to rotation. The gravitational moment is related to a dimensionless measure of the strength of rotation. In general, we expect that

\[ J_2 = \frac{q}{2}; \quad \varepsilon = -\frac{5q}{6} \]

with the \( n=0 \) term dominating. For example, \( \Lambda_{2,0} = 0.5 \) for a Maclaurin spheroid (the technical name for the uniform density case we studied here) and \( \Lambda_{4,0} = -0.536 \).

It is plausible, and turns out to be actually true that these coefficients are diagnostic of the density structure. For example,

\[ \Lambda_{2,0} = \left( \frac{5}{\pi^2} - \frac{1}{3} \right) = 0.173 \]

for the case of the model we studied for Jupiter where \( P = K \rho^2 \).

**The Radau-Darwin Approximation**

There is an approach, conceptually the same as described above, that works for bodies that are close to constant density. This is not a bad approximation for terrestrial planets and icy satellites but is rather poor for giant planets. In this approach, you write the total gravitational potential in the form

\[ U = U_s + \int_0^\rho \left( \frac{dP}{d\rho'} \right) \left( \frac{d\rho}{d\rho'} \right) \]

where this form comes from using hydrostatic equilibrium: \( \nabla P = \rho \nabla U \). The subscript “s” refers to some (arbitrary) reference surface. This form of the potential must also be identical to that derived from the fundamental formula for gravity, i.e.

\[ U \equiv W + G \int \frac{\rho(r') d^3 r'}{|\mathbf{r} - \mathbf{r}'|} \]
where $W$ is the “external” potential (tides or rotation). One then assumes that the equipotential surfaces inside the planet can be written in the form of $r = r_0(1 + \varepsilon(r)P_2 + \ldots)$ and derives a messy integrodifferential equation for the functions $\varepsilon(r)$. To make a long story short (look at Hubbard’s book) it turns out that in place of the simple previous result ($\Lambda_{2,0} = 0.5$), we obtain 

$$\frac{C}{Ma^2} = \frac{2}{3^2 \Lambda_{2,0}^5}$$

(This is called the Radau-Darwin approximation. It is not merely an approximate theory; it is also an approximation to that theory.) Recall that at this level of approximation, $\Lambda_{2,0} = J_2/q$. Here are some values that this formula predicts:

<table>
<thead>
<tr>
<th>$\Lambda_{2,0}$</th>
<th>$C/Ma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.4</td>
</tr>
<tr>
<td>0.45</td>
<td>0.383</td>
</tr>
<tr>
<td>0.40</td>
<td>0.366</td>
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<tr>
<td>0.35</td>
<td>0.347</td>
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<tr>
<td>0.3</td>
<td>0.326</td>
</tr>
<tr>
<td>0.25</td>
<td>0.303</td>
</tr>
</tbody>
</table>

What are the Data?

Bodies fall into two classes: those for which hydrostatic effects dominate (i.e., “rapid rotators”) and those for which the rotational bulge is no bigger than the other effects on topography and gravity. The rapid rotators further subdivide into those for which Radau-Darwin is roughly valid (terrestrial bodies and icy satellites) and the gaseous bodies (in which the density variations are too large, and a more complex and detailed theory has been devised - one example is the solution given above for $P \propto \rho^2$. )
<table>
<thead>
<tr>
<th>Body</th>
<th>Measured $J_2$</th>
<th>Measured $q$</th>
<th>$J_2/q$</th>
<th>Inferred $C/\text{Ma}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>$(8\pm6)\times10^{-5}$</td>
<td>$1\times10^{-6}$</td>
<td>~1</td>
<td>*</td>
</tr>
<tr>
<td>Venus</td>
<td>$(6\pm3)\times10^{-6}$</td>
<td>$6.1\times10^{-8}$</td>
<td>~$10^2$</td>
<td>*</td>
</tr>
<tr>
<td>Earth</td>
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<td>$3.5\times10^{-3}$</td>
<td>0.31</td>
<td>0.33</td>
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<tr>
<td>Moon</td>
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<td>$7.6\times10^{-6}$</td>
<td>~30</td>
<td>*</td>
</tr>
<tr>
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<td>$4.6\times10^{-3}$</td>
<td>0.43</td>
<td>0.375</td>
</tr>
<tr>
<td>Jupiter</td>
<td>$1.4733\times10^{-2}$</td>
<td>0.089</td>
<td>0.166</td>
<td>†</td>
</tr>
<tr>
<td>Saturn</td>
<td>$1.646\times10^{-2}$</td>
<td>0.153</td>
<td>0.107</td>
<td>†</td>
</tr>
<tr>
<td>Uranus</td>
<td>$3.352\times10^{-3}$</td>
<td>0.035</td>
<td>0.096</td>
<td>†</td>
</tr>
<tr>
<td>Neptune</td>
<td>$3.538\times10^{-3}$</td>
<td>0.028</td>
<td>0.125</td>
<td>†</td>
</tr>
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<tr>
<td>Callisto</td>
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<td>$0.037\times10^{-3}$</td>
<td>0.35</td>
<td>0.35</td>
</tr>
</tbody>
</table>

*Non-hydrostatic (at this low level of rotation) so method does not work.

†Hydrostatic but Radau -Darwin doesn’t work. More complex theory can however constrain moment of inertia (see next chapter for results). Note that the result for Jupiter does agree quite well with the exact prediction of $\Lambda_{2,0} = 0.173$ for $P \propto \rho^2$. $J_4$ and even $J_6$ are used for these planets to improve the estimates of internal structure.

1In the case of the Galilean satellites, there are tidal and rotational bulges. The inversion of the Galileo spacecraft data assumes that these are related hydrostatically. The values quoted above for $J_2$ are not what you will find in the published papers (see references in Showman and Malhotra, *Science*, Oct 1, 1999). The reason is that they (i.e., Anderson et al) used a different normalization for their gravity field coefficients than the (more common) one I used above. To find the usual $J_2$ you have to take the published $C_{22}$ and multiply by $4/3$. 