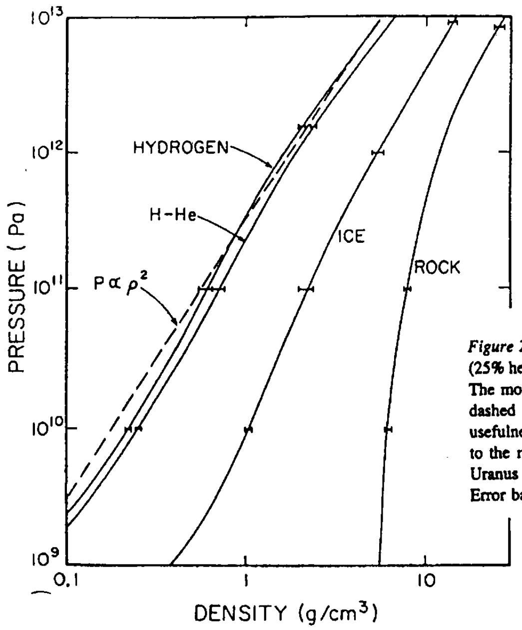
Ge 131
Chapter 7
Generic Planetary Models

Nature of the Equations of State

At sufficiently low pressures in solids, the density of a material is little affected by pressure and $M \propto \rho R^3$ with $\rho \sim$ constant. You then expect that the radius of an object will vary as the cube-root of mass:

$$R \propto M^{1/3}$$

This makes sense for the low mass behavior if the material has a finite density as the pressure goes to zero. Recall that the low mass behavior requires $P \ll K$, where K is the bulk modulus. Since $P \sim R^2$ and $\rho \sim \rho_o(1+P/K)$ where ρ_o is the zero-pressure density (chapter 2 and homework #1), the mean density of a planet will deviate quadratically in R from the zero pressure density as the radius of the planet increases. However, this makes no sense for any planet where hydrogen and helium are significant, because these materials can expand without bound as the pressure is lowered at finite temperature. (Equivalently, P/K=1 in ideal gases, even if P is very small). Even materials that expand to a finite density as $P \rightarrow 0$ will begin to deviate from the simple cube root behavior, at sufficiently high pressure.



Stevenson, Ann. Rev. Earth & Planet. Sci. (1982)

Figure 2 Typical isentropes relevant to the giant planets. The hydrogen and hydrogen-helium (25% helium by mass) isentropes pass through $T = 170$ K at $P = 10^5$ Pa (appropriate to Jupiter). The molecular-metallic hydrogen transition has been smoothed but may be discontinuous. The dashed line is the best fitting curve of the form $P = K\rho^2$ for pure hydrogen, indicating the usefulness of this simple analytic approximation. The isentropes of "ice" and "rock" correspond to the mixtures used by Hubbard & MacFarlane (1980) and are chosen to be appropriate for Uranus (i.e. $T \simeq 2000$ K at $P = 2 \times 10^{10}$ Pa for ice, $T \simeq 7000$ K at $P = 6 \times 10^{11}$ Pa for rock). Error bars indicate the uncertainties.

Temperature (e.g. whether the body is adiabatic) also has a significant effect.

Polytropes

Notice that the figure above shows equations of state that are very roughly straight lines on a log-log plot. This suggest a power law relationship. A polytrope is defined to be an equation of state in such a form: $P \propto \rho^n$. This will have relevance for materials that expand without limit as $P \rightarrow 0$, or bodies that are so massive that the mean density is much larger than the zero pressure density. Since the central pressure according to hydrostatic equilibrium must scale as $\langle\rho\rangle gR \sim (M/R^3)(GM/R^2)R \sim GM^2/R^4$ and the density scales as M/R^3, we immediately have

$$\frac{M^2}{R^4} \propto \left[\frac{M}{R^3}\right]^n \quad \Rightarrow \quad R \propto M^{\frac{2-n}{4-3n}}$$

Notice three things:

(i) We recover the result $R \propto M^{1/3}$ in the limit $n \rightarrow \infty$; this makes sense since that limit is incompressible material (infinitesimal density change gives large pressure change).

(ii) We get $R \propto M^{-1/3}$ when $n=5/3$. Recall that this is the case for an ideal Fermi gas. It is therefore applicable to superJupiters and white dwarfs. At sufficiently high mass (but still non-relativistic) it applies to all materials irrespective of atomic mass. *At sufficiently high mass, all degenerate bodies become smaller as you add mass to them.*

(iii) We get R independent of mass when $n=2$; this is approximately relevant to Jupiter and Saturn.

(iv) We get no sensible result when $n=4/3$. Although it is not obvious without further analysis, this turns out to be a hint of instability and the origin of the Chandrasekhar limit to massive stars, since $n=4/3$ corresponds to the ideal Fermi gas limit for relativistic electrons. (It also corresponds to the limit where radiation pressure dominates) .

There is a large literature on polytropes and you can find out all about them in Chandrasekhar's book on stellar structure. In general, the equation of hydrostatic equilibrium does not have an analytical solution, but n=2 is a special case since it leads to a linear differential equation. Let's derive this, since it is of practical use. We assume $P = K\rho^2$:

$$\frac{dp}{dr} = 2K\rho(r)\frac{d\rho}{dr} = -\rho(r)g(r)$$

$$\therefore 2K\frac{d\rho}{dr} = -g(r) = -\frac{G}{r^2}\int_0^r 4\pi x^2\rho(x)dx$$

$$\therefore \frac{d}{dr}(r^2\frac{d\rho}{dr}) = -k^2r^2\rho; \quad k^2 \equiv \frac{2\pi G}{K}$$

$$\therefore \frac{d^2}{dr^2}(r\rho) = -k^2.(r\rho)$$

$$\therefore \rho(r) = A\frac{\sin(kr)}{kr} + B\frac{\cos(kr)}{kr}$$

$$A = \rho_c; \quad B = 0$$

$$kR = \pi \quad \Rightarrow R = \sqrt{\frac{\pi K}{2G}}$$

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 3\int_0^1 x^2.\rho_c\frac{\sin(\pi x)}{\pi x}dx = \frac{3}{\pi^2}\rho_c$$

Note that B=0 because we must have non-divergent density at the origin. Most important, we get an explicit formula for the radius, and it is independent of mass, as promised. For the realistic choice of $K=2.1 \times 10^{12}$ cgs (a cosmic hydrogen/helium mixture), this formula gives a radius of 70,300 km. The mean radius of Jupiter is 69,800km. The inferred central density is $\pi^2/3$ times the mean density, corresponding to 4.38 g/cc and a pressure of 40 Megabars. The radius thus obtained should apply equally well to Saturn, but the observed radius of Saturn is only ~58,000km. *The fact that Saturn is smaller than Jupiter must be because it has heavier constituents, not because it has lower mass.*

One way to think about the effect of adding heavier stuff (uniformly) to hydrogen is that it affects the density but not the pressure. In that (crude) way of thinking, we can think of the effective K as the value appropriate for hydrogen divided by $(1+y)^2$ where y is the mass fraction of heavier stuff. In that way, $P=K_{eff}\rho^2$ will still apply. So then the radius is reduced by 1+y and Saturn would require that y~0.2, implying ~15 Earth masses of heavier stuff. The same amount of heavy stuff in Jupiter is a smaller fraction of the total mass ($y_{Jupiter}$~0.05) and permitted by the observed radius! Detailed models allow this, though the uncertainties remain large (see the Oct. 1, 1999 issue of *Science.*) The presence of a core (as distinct from just heavy element enrichment) is unresolved, though likely.

Mass-Radius Relationships

In the first figure (below) we see that hydrogen-helium adiabatic bodies have roughly constant radius (as promised) but actually expanding as they approach low mass ideal gas adiabatic behavior ($P \propto \rho^{1.45}$). The radius actually declines as you go to still higher masses (the brown dwarf regime) though the effect is modest if (as is usually the case) these bodies are also hotter and thus less close to the degenerate limit given by the solid lines. This figure also shows us that Uranus and Neptune do not have a simple interpretation.

Mass-Radius Relationships

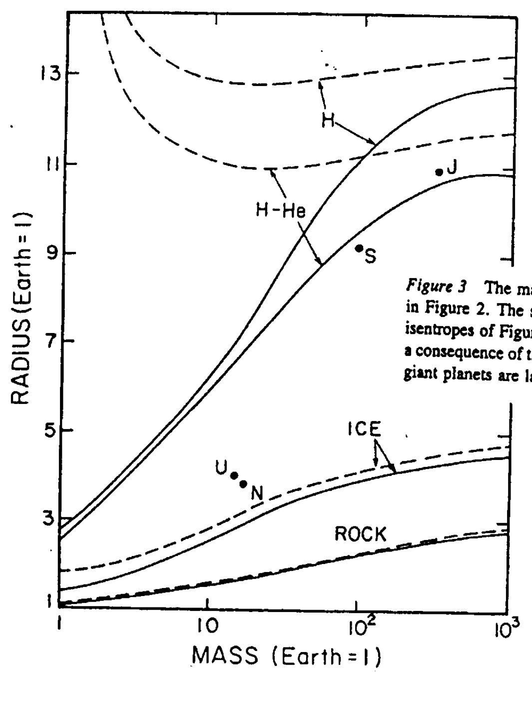


Figure 3 The mass-radius relationship for self-gravitating bodies of the same compositions as in Figure 2. The solid lines are for cold matter (T = 0 K); the dashed lines correspond to the isentropes of Figure 2. The insensitivity of radius or mass for hydrogen and hydrogen-helium is a consequence of the approximate validity of $P \propto \rho^2$ (see text for discussion). The positions of the giant planets are labelled by J, S, U, and N.

Stevenson (1982)

In the second figure we see the much less dramatic effects in low mass objects. But even here, one sees that the difference in mean density between Ganymede and (say) Dione is in major part due to ice compression (more precisely, phase changes) rather than intrinsic compositional difference.

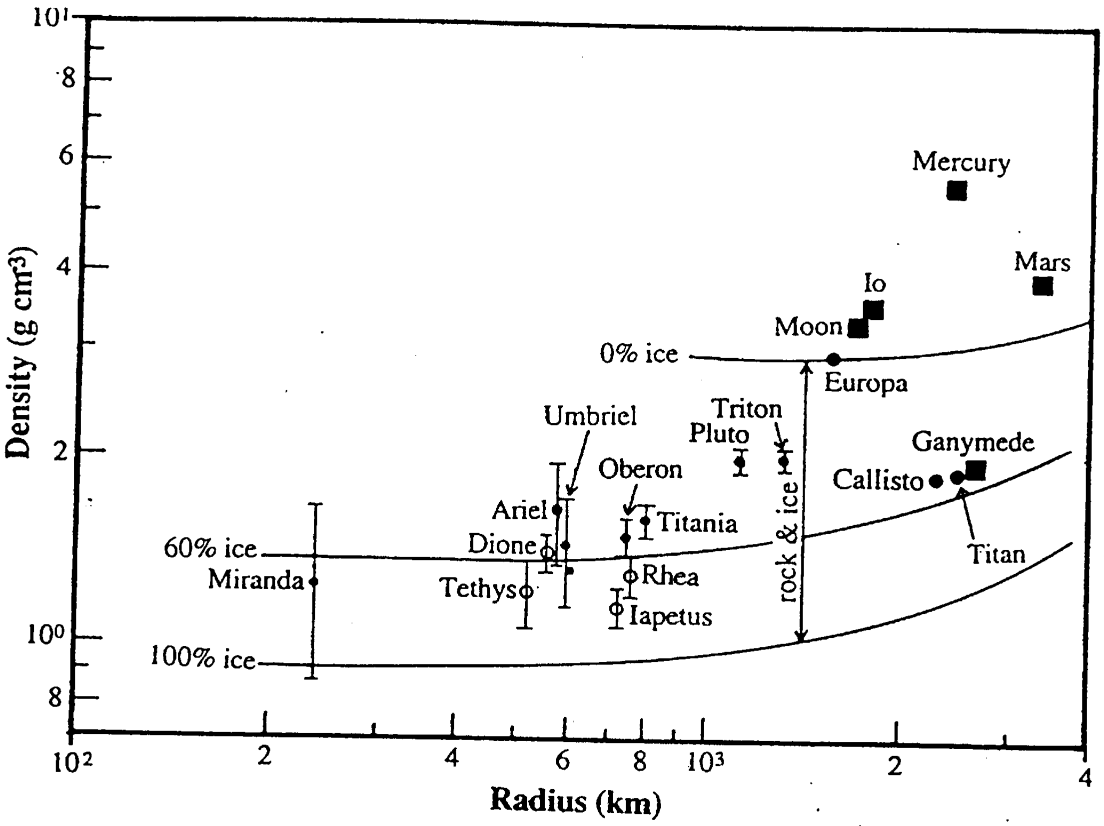


Figure VI.30 Densities of the satellites of the outer planets. A solar-proportion mixture of water ice and rock (60%), with an uncompressed density of 1.3 to 1.5, could be reconciled with many of these data. The Saturnian satellites show a wide spread of densities without any clear radial trend.

The Virial Theorem

The gravitational energy of a planet can be written

$$E_G = -\int_0^M \frac{Gmdm}{r(m)}$$

But we also have the hydrostatic equation, which together with the definition of m(r) leads to the following:

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}$$
$$dm = 4\pi r^2 \rho dr$$
$$\Rightarrow \frac{dp}{dm} = -\frac{Gm}{4\pi r^4}$$

Looking back at the gravitational energy, we see that

$$E_G = \int_0^M 4\pi r^3 dP = 3\int_0^M \{d[\frac{4}{3}\pi r^3.P] - P.dV\}$$
$$\Rightarrow E_G = -3\int_0^M \frac{P}{\rho} dm \quad [\text{or } -3\int PdV]$$

This last result is called the Virial theorem. Note it requires that P.($4\pi r^3$ /3) be zero at both the center and outer surface of the planet, which is true since r=0 and P=0 respectively , at those locations.

We can now consider a perturbation to the planet in which hydrostatic equilibrium is preserved, but there are infinitesimal changes in density and pressure at each mass element:

$$\delta E_G = -\int_0^M Gm.\delta(1/r).dm = \int_0^M \frac{Gm}{r^2}.\delta r.dm$$
$$= -\int_0^M 4\pi r^2.\frac{dP}{dm}.\delta r.dm = -\int_0^M \delta(\frac{4}{3}\pi r^3).\frac{dP}{dm}.dm$$
$$= -\int_0^M \{d[\delta(\frac{4}{3}\pi r^3).P] - \delta[\frac{d(4\pi r^3/3)}{dm}].P.dm\}$$
$$\therefore \delta E_G = \int_0^M P.\delta(1/\rho).dm$$

So the change in gravitational energy equals the work done on the sample. (This interpretation only works if you think about constant composition. Obviously you can also lower the energy by moving the denser stuff to higher pressures and the less dense stuff to lower pressures (i.e., δ(1/ρ) is negative where P is high and positive where P is low).

The Energy Output of Planets

We can use this last result to derive a very important result concerning energy balance. The "luminosity" (total energy output) of a planet comes from explicit energy sources (e.g. radioactive decay, tidal heating), here labeled Q (per unit mass) but can also come from changes in internal and gravitational energy:

$$L = \int_0^M Q dm - \frac{d(E_{\text{int}} + E_G)}{dt}$$

Consider a planet that is not differentiating (i.e., not changing the distribution of constituents). Now, planets are degenerate bodies and we can thus conveniently subdivide the internal energy and pressure into zero temperature pieces and finite temperature corrections in the form:

$$E_{\text{int}} \approx E_0 + C_V T$$
$$P \approx P_0 + \gamma \rho C_V T$$
$$P_0 = -\frac{dE_0}{d(1/\rho)}$$

(where the approximation signs can be easily dispensed with by writing more general expressions of the thermal correction; it doesn't matter). So:

$$\frac{d(E_{\text{int}} + E_G)}{dt} = \int_0^M \{\frac{dE_0}{dt} + C_V \frac{dT}{dt} + P_0 \frac{d(1/\rho)}{dt} + \gamma \rho C_V T \frac{d(1/\rho)}{dt}\} dm$$

But the first and third terms in the integral cancel by the Virial theorem, and the last term is small because we can estimate that $d(1/\rho)/dt = (\alpha.dT/dt)/\rho$, where α is the coefficient of thermal expansion, and $\alpha T \ll 1$. So we finally get, to an excellent approximation,

$$L = \int_0^M [-C_V \frac{dT}{dt} + Q] dm$$

What this means is that as a planet cools and contracts, the gravitational energy becomes more negative and the dominant part of the internal energy (the part due to compression) becomes more positive, but that these almost exactly cancel! The consequence is that the dominant energy output associated with the planet evolution (aside from radioactivity and external sources of heating) is the change in the thermal energy. This seems intuitively obvious but it is nonetheless widely misunderstood (e.g., there are books which say that Jupiter's energy output is derived from contraction when in fact the dominant effect is simply cooling. Of course, cooling *implies* contraction, but the energy available is nonetheless thermal).

Note, however, that this is not <u>exactly</u> true, and <u>very importantly</u> does not include gravitational energy release due to compositional changes (e.g. core formation, differentiation in general). This can be immediately recognized by noting that changes in density (determining changes in gravitational energy) do not only come about from work done by pressure... they can also arise from moving constituents around.

<u>*The equation derived here for luminosity is very different from non-degenerate bodies!*</u> A non-degenerate star that loses heat as it contracts is nonetheless getting hotter internally, i.e. it behaves as though it has negative thermal capacity. You can see this by returning to the original form of the Virial theorem and noticing that the gravitational energy is larger than the internal energy, yet scales as T and is negative, so the luminosity is proportional to +dT/dt rather than -dT/dt. (This was profoundly puzzling in the 1920's, the early days of stellar structure theory and before degeneracy was understood.) Of course, the ability of a star to heat up as it contracts and loses heat was crucial for understanding thermonucleosynthesis.