Lecture 14

Waves

14.1 Two Masses on a string

When we set out to solve the diffusion equation, we first solved the simpler case of solving for temperatures in a few connected pots of water, and then we extended the results to a continuous temperature distribution in a rod. Our goal for this lecture is to determine the equation of motion for waves on a string. But, like the case for diffusion, we will first solve the discrete equation of 2 masses on an otherwise massless string. This equation is solved in a manner analogous to the solution to the coupled springs that we examined previously.

Consider the situation illustrated in Figure 14.1. Two masses are connected by a massles string to each other and to the walls. The strings have a tension (which is a force) of T. At rest, then, each mass feels a force T pulling in both directions, so the masses stay at rest. If one of the masses is raised, however, a new downward force is felt. Now, in real life, part of the new force will be a spring-like force caused by the fact that the string must be stretching and must have some spring-like restoring constant, but we are going to consider very very small motions so that these forces are miniscule compared to the other, more important, restoring forces. What are these other restoring forces? These are the components of the tension T in the direction x (see Figure 14.1).

When mass 1 is moved by distance x_1 up above its zero position, it feels a restoring force of $T \sin\theta$, where θ is the angle that the string makes with the wall. With l the distance of the mass from the wall,

$$\tan\theta = \frac{x_1}{l}$$

or, because the distance along the string is now $\sqrt{x_1^2 + l^2}$,

$$\sin\theta = \frac{x_1}{\sqrt{x_1^2 + l^2}}$$

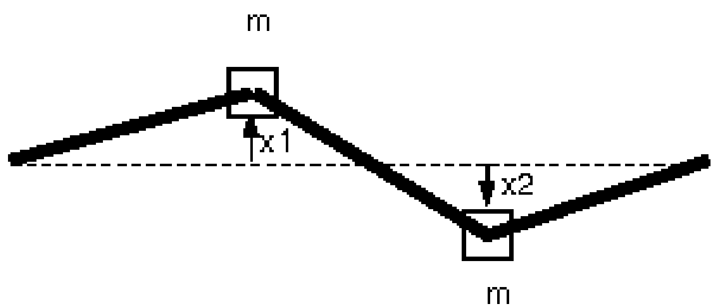

Figure 14.1: Two masses on a massless string.

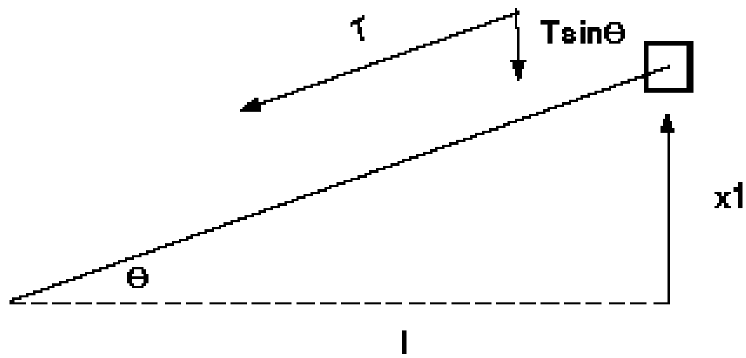

Figure 14.2: The restoring force for the string tension problem.

Now, in our case, we $x_1 \ll l$, the motions are very small compared to the other lengths in the system. To first order, then,

$$\sin\theta = \frac{x_1}{l},$$

which will certainly simplify things.

We can now write down the two coupled equations for the two masses:

$$m\ddot{x}_1 = -\frac{T}{l}x_1 - \frac{T}{l}(x_1 - x_2)$$

$$m\ddot{x}_2 = -\frac{T}{l}x_2 - \frac{T}{l}(x_2 - x_1)$$

or, in our favored form,

$$\ddot{x}_1 + 2\frac{T}{lm}x_1 - \frac{T}{lm}x_2 = 0$$
$$\ddot{x}_2 + 2\frac{T}{lm}x_2 - \frac{T}{lm}x_2 = 0.$$

We do our usual thing of assuming that the solutions are

$$x_1 = B_1 \exp(i\omega t)$$
$$x_2 = B_2 \exp(i\omega t)$$

and when we substitute in to the equations and divide by $\exp(i\omega t)$ we get

$$-\omega^2 B_1 + 2\frac{T}{lm}B_1 - \frac{T}{lm}B_2 = 0$$
$$-\omega^2 B_2 + 2\frac{T}{lm}B_2 - \frac{T}{lm}B_1 = 0$$

or, in matrix form again,

$$\begin{pmatrix} -\omega^2 + 2\frac{T}{lm} & \frac{-T}{lm} \\ \frac{-T}{lm} & -\omega^2 + 2\frac{T}{lm} \end{pmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0$$

We know that, for a non-trivial solution to the equations, the determinant of this matrix must now be zero, so we have

$$(-\omega^2 + 2\frac{T}{lm})^2 - (\frac{T}{lm})^2 = 0$$

which we can quickly factor to be

$$(\omega^2 - 3\frac{T}{lm})(\omega^2 - \frac{T}{lm}) = 0,$$

so the two characteristic frquencies of the problem are

$$\omega_1 = \pm\sqrt{\frac{T}{lm}}$$
$$\omega_2 = \pm\sqrt{\frac{3T}{lm}}$$

If we want to determine the eigenmodes of these eigen frequencies, we can substitute back in above. For ω_1 we get

$$B_1(\frac{T}{lm}) - B_2\frac{T}{lm} = 0$$
$$B_2(-\frac{T}{lm}) + B_2\frac{T}{lm} = 0$$

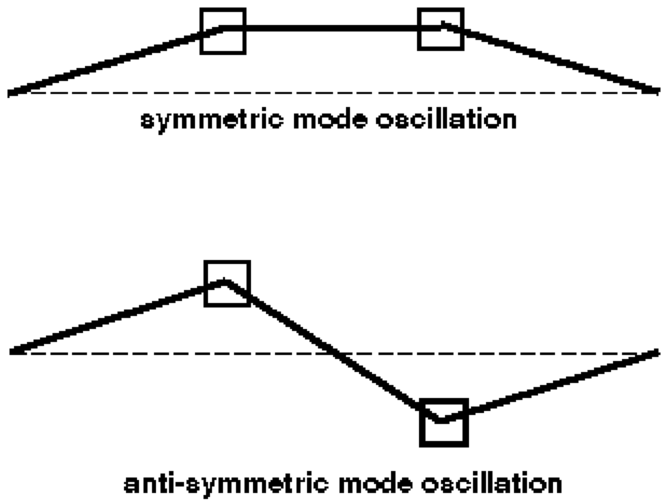

Figure 14.3: The two eigenmode oscillations of two masses on a string.

or

$$B_1 = B_2$$

which is, again, a symmetric mode vibration (Figure 14.1).

For ω_2 we get:

$$\begin{aligned} B_1(-\frac{T}{lm}) - B_2\frac{T}{lm} &= 0 \\ B_1(-\frac{T}{lm}) - B_2\frac{T}{lm} &= 0 \end{aligned}$$

or

$$B_1 = -B_2$$

which is the anti-symmetric mode vibration.

14.2 More masses

We eventually want to extend to the case of a string with a particular mass per unit length, which is the limiting case of an infinite number of tiny masses separated by an infinitely small distance. On our way to working there, we will first just increase the number of masses by one. For the three mass case, we have a simple extension of the two mass case (and defining $\omega_0 = \sqrt{\frac{T}{ml}}$), and the equations of motion are obviously

$$\begin{aligned}
\ddot{x}_1 &= -\omega_0^2 x_1 - \omega_0^2(x_1 - x_2) \\
\ddot{x}_2 &= -\omega_0^2(x_2 - x_1) - \omega_0^2(x_2 - x_3) \\
\ddot{x}_3 &= -\omega_0^2 x_3 - \omega_0^2(x_3 - x_2)
\end{aligned}$$

Skipping the steps we are now over-familiar with, we find that for our favorite solutions to work, we must have

$$\begin{vmatrix} (-\omega^2 + 2\omega_0^2) & -\omega_0^2 & 0 \\ -\omega_0^2 & (-\omega^2 + 2\omega_0^2) & -\omega_0^2 \\ 0 & -\omega_0^2 & (-\omega^2 + 2\omega_0^2) \end{vmatrix} = 0$$

which expands to

$$(-\omega^2 + 2\omega_0^2)^3 - 2(-\omega^2 + 2\omega_0^2)\omega_0^4 = 0$$

or

$$(-\omega^2 + 2\omega_0^2)[(-\omega^2 + 2\omega_0^2)^2 - 2\omega_0^4] = 0$$

which becomes

$$(-\omega^2 + 2\omega_0^2)(\omega^2 - \omega_0^2(2 - \sqrt{2}))(\omega^2 - \omega_0^2(2 + \sqrt{2})) = 0$$

which gives the three characteristic frequencies

$$\begin{aligned}
\omega_1 &= \pm\omega_0\sqrt{2} \\
\omega_2 &= \pm\omega_0\sqrt{2 + \sqrt{2}} \\
\omega_3 &= \pm\omega_0\sqrt{2 - \sqrt{2}}
\end{aligned}$$

Now to determine the characteristics modes. For ω_1 we have

$$\begin{aligned}
B_2 &= 0 \\
-\omega_0^2 B_1 - \omega_0^2 B_3 &= 0 \\
B_2 &= 0
\end{aligned}$$

which is just the anti-symmetric mode with the center mass stationary (Figure 14.2).

For ω_3 we have

$$\begin{aligned}
B_1\omega_0^2\sqrt{2} - \omega_0^2 B_2 &= 0 \\
-B_1\omega_0^2 + B_2\omega_0^2\sqrt{2} - B_3\omega_0^2 &= 0 \\
-\omega_0^2 B_2 + B_3\omega_0^2\sqrt{2} &= 0
\end{aligned}$$

This case is not a simple one with which we are familiar, but we can figure it out quickly. From the first equation, we get

$$B_2 = \sqrt{2}B_1$$

which we can substitute into the second equation to get

$$B_1 = B_3$$

This motion is one where the two outer masses oscillate symmetrically and the inner mass follows them, only further.

For ω_2 we get

$$\begin{aligned}
-B_1\omega_0^2\sqrt{2} - \omega_0^2 B_2 &= 0 \\
-B_1\omega_0^2 - B_2\omega_0^2\sqrt{2} - B_3\omega_0^2 &= 0 \\
-\omega_0^2 B_2 - B_3\omega_0^2\sqrt{2} &= 0
\end{aligned}$$

Using the same method, we get

$$B_2 = -\sqrt{2}B_1$$

and

$$B_1 = B_3$$

which is very similar to the ω_1 mode except that the middle mass is moving out of synch with the end masses.

14.3 The wave equation

Now we jump to strings. Had you been thinking hard while reading the last section, you might have had a brilliant insight that what we were really doing was taking second derivatives the whole time. If you haven't been thinking very hard then maybe you will be convinced that it is true after seeing the following derivation.

Consider now a string of total length L, tension T, and mass per unit length p. The equation of motion for the upward displacement, u of a tiny section of string at position x, which is separated by another tiny section of the string by distance Δx is

$$(\Delta x p)\ddot{u}(x) = -\frac{T}{\Delta x}[(u(x) - u(x - \Delta x)) - (u(x) - u(x + \Delta x))]$$

If we take this equation in the limit that $\Delta x \to 0$ we get the familiar second derivative for x and the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{p}\frac{\partial^2 u}{\partial x^2}$$

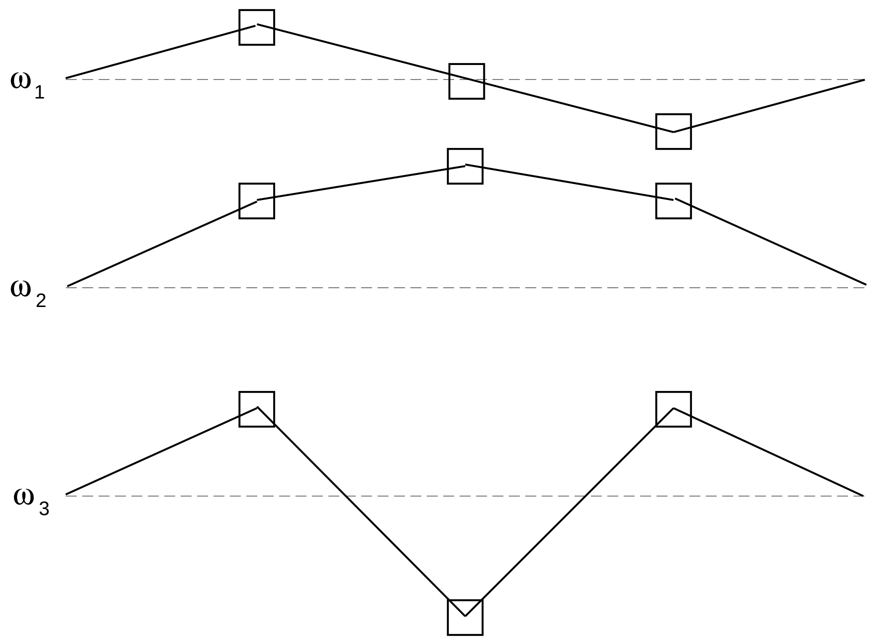

Figure 14.4: The modes for 3 masses on a string.

We will suggestively define $v = \sqrt{\frac{T}{p}}$ and rewrite the equation as the classical one-dimensional *wave equation:*

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

How do we solve this equation? This thing is very similar to the diffusion equation. And, if you recall, for the diffusion equation we found that there were a huge variety of solutions and we just picked a good one that worked for whatever initial conditions we had. The most useful solution to the wave equation turns out to be out the easiest to discover, too, so we are in luck. Just like for the diffusion equation, we will seek a solution where the spatial and time parts of the function are independent or *separable*, which means that we can generally write

$$u(x,t) = A(x)B(t)$$

which we can now plug into the wave equation to get

$$A\frac{d^2 B}{dt^2} = v^2 \frac{d^2 A}{dx^2} B$$

We then separate these into
$$\frac{A}{\frac{d^A}{dx^2}} = v^2 \frac{B}{\frac{d^2B}{dt^2}}$$
Once again, we assert that for this to be true for all places and all times that the two sides of this equation must indepently be equal to a constant, which we will again call λ^2. We therefore have two equations:

$$\begin{aligned}
\frac{d^2A}{dx^2} &= \lambda^2 A \\
\frac{d^2B}{dt^2} &= \lambda^2 v^2 B
\end{aligned}$$

for which we know the solutions to be

$$\begin{aligned}
A &= A_1 \exp(\lambda x) + A_2 \exp(-\lambda x) \\
B &= B_1 \exp(\lambda vt) + B_2 \exp(\lambda vt)
\end{aligned}$$

We can pick any values of λ that we like (or that fit the initial conditions, actually), and, to make life interesting and to get an oscillating solution, we will pick $\lambda = ik$ where k is now defined to be real. We therefore get a general solution of

$$u(x,t) = u_0 \exp[i(kx - kvt)] + u_1 \exp[i(kx + kvt)]$$

This solution is one of traveling waves on an infinite string. Figure 14.3 shows solutions for $u_1 = 0$, the positive travelling wave, and for u_0=0, the negative traveling wave. To truly make the solution general, we should allow arbitrary phase, too, but we will ignore that added complication. The figure also shows the effect of increaseing v: the waves travel faster! An infinite string will support waves of any wavelength; the velocity of the wave, though, remember, is controlled by the tension and mass of the string.

What if we have equal power in the positive and negative traveling waves ($u_0 = \pm u_1$)? We should get a wave that doesn't go anywhere, or a *standing wave.* The solution for such a wave is illustrated in Figure 14.3.

14.4 Finite strings

What if we take our infinite string, lop off two ends, and attach the string to walls? The same solutions to the wave equation must work, but we now have to use solutions that fit the initial conditions. Whatever initial conditions there are, they all must satisfy the conditions that $u(0,t) = u(l,t) = 0$ where 0 and l are the two sides of the wall attached. These conditions then require that

$$\begin{aligned}
u(0,t) = u_0 \cos(kvt) + u_1 \cos(kvt) &= 0 \\
u(l,t) = u_0 \cos(kl - kvt) + u_1 \cos(kl + kvt) &= 0
\end{aligned}$$

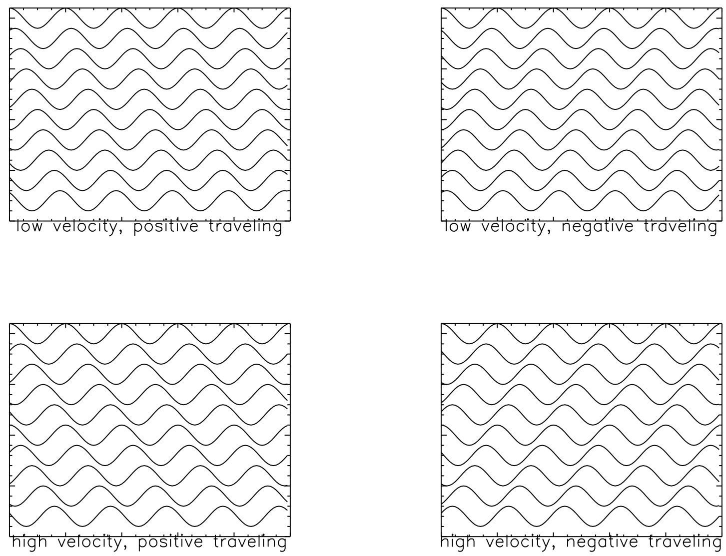

Figure 14.5: Travelling wave solutions to the wave equation with an infinite string.

Using our usual arguments, we could say that to be satisfied for all times, we must have the conditions that $u_0 = u_1 = 0$. But, we say with a scorn, this is such a trivial solution that it is not worth our wasting our precious time. What about real, important, interesting solutions? Just like before, we can assert that such interesting solutions only work if we can force the determinant in the denominator of Cramer's rule to be zero,

$$\begin{vmatrix} \cos(kvt) & \cos(kvt) \\ \cos(kl - kvt) & \cos(kl + kvt) \end{vmatrix} = 0$$

which becomes

$$\cos(kvt)\cos(kl + kvt) - cos(kvt)cos(kl - kvt) = 0$$

from which we can factor out the $\cos(kvt)$, which can never be zero at all times, to get

$$\begin{aligned} \cos(kl + kvt) - \cos(kl - kvt) &= 0 \\ \cos(kl)\cos(kvt) - \sin(kl)\sin(kvt) - \cos(kl)\cos(kvt) - \sin(kl)\sin(kvt) &= 0 \\ \sin(kl)\sin(kvt) &= 0 \end{aligned}$$

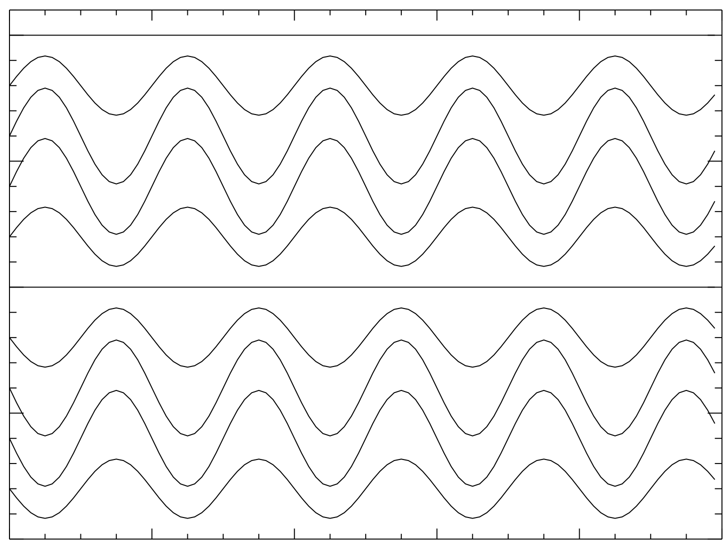

Figure 14.6: Standing waves.

Which means that we must have either

$$\sin(kvt) = 0$$

for all t, which imples $k = 0$ which is, again, a trivial solution, or

$$\sin(kl) = 0$$

which is the solution for which we've been looking, for it implies that

$$k = \pm\frac{n\pi}{l}, \quad n = 0, 1, 2, 3...$$

and, for, any of these solutions, it must be that $u_0 = -u_1$, which implies that there is equal power in the positive traveling and the negative traveling component of the wave, implying a standing wave. But note, now, that the standing wave can have only a very few set of specific wavelenghts ($\lambda = 2\pi/k = 2l/n$).

Let's look at the first few of those solutions (Figure 14.4). For the $n = 1$, $\lambda = 2l$ case, we simply have half of a sine wave. As we increase, n, we get more halves of the sine wave, such that there are n peaks (or troughs) in the curve. These modes of vibtration of the string are called *harmonics* or *overtones* of the

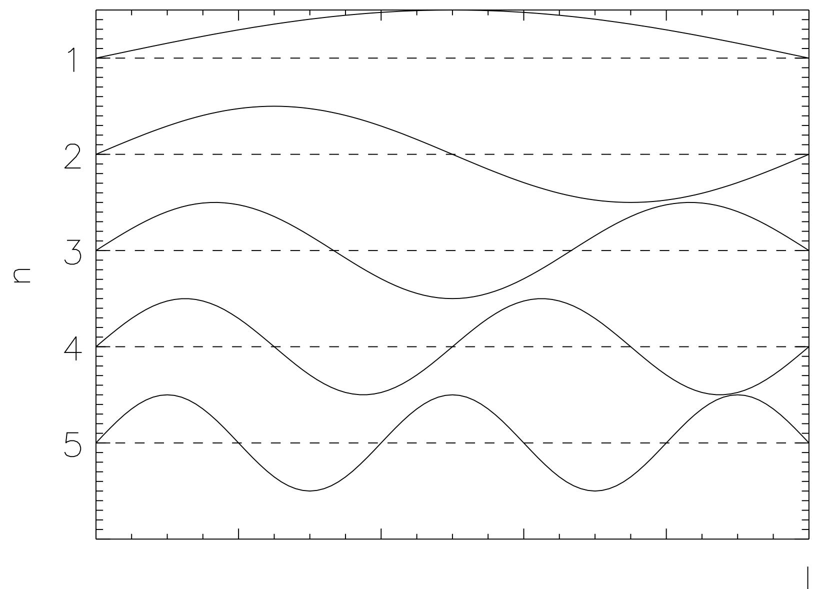

Figure 14.7: The first 5 harmonics of an oscillating string.

string. The frequency of vibration of these harmonics is $\nu = v/\lambda = vn/2l$, thus the vibration frequencies are multiples of the lowest, or *fundamental* frequency of vibration.

Now, as any musician will tell you, a factor of two in frequency of vibration is a full octave of a musical scale (i.e. middle A is 440 Hz, and A an octave up is 880 Hz, and an octave higher, still is 1760 Hz). So this tells us that a string will vibrate in such a way as to only allow the same note, but in different octaves, to be heard.

The frequency of a plucked string is, again $\nu = vn/2l$, where v is the wave velocity, which is given by $v = \sqrt{T/p}$ (recall that T is the tension and p is the mass per unit length). So what if we have a guitar, for example, where the strings are all the same length, and we want the fundamental of some of the strings to be higher than that of others? We increase the mass of the strings. The lowest tones, which are the lowest frequencies, have the heaviest strings, while the highest tones have light thin strings. For fine tuning of the notes, the tension of the string can be adjusted. In principle, the tension could be used to control everything, and all strings could be identical, but strings only work well under a small range of tensions, so changing the masses is more practical. To make much much lower notes, we could either make very heavy strings, which would be too hard to play, or, instead, increase the length. So bass violins are larger than cellos are larger and violas are larger than violins so that their strings harmonic vibrations are at progressively lower tones.

Strings will vibrate in all of the infinite number of modes available to it. So how come when you pluck the string of an instrument, you only hear one note? It is because, if you do a Fourier decomposition of the initial conditions (say that you have grabbed the string at the center and pulled it straight up) you would find that you have most of the power in the fundamental. Initial conditions that put more power in the higher harmonics would lead to different tones being heard. With some practice, one can excite the strings in a guitar at these different harmonics intead of at the fundamental.