# Lecture 13

# Coupled Oscillations

## 13.1 Coupled springs

Understanding coupled oscillators is crucial to understanding motions of waves, which are vital for understanding the interior of the earth, without which none of us would be alive. Coupled oscillators are therefore an integral part of our lives.

Consider the series of coupled springs, similar to those for which you obtained a numeric solution in the problem set, illustrated in Figure 13.1. In equilibrium, the two masses, $m_1$ and $m_2$, are at their rest positions, $x_1 = 0$ and $x_2 = 0$. When displaced from their equilibria, the three springs, with spring constants $k_1$, $k_{12}$, and $k_2$, are stretched by $x_1$, $x_2 - x_1$, and $x_2$. The equation of motion for the first mass (with $m_1 = m$) is (and we are now going to use "dot" notation, where $\dot{x} = \frac{dx}{dt}$

$$m\ddot{x}_1 = -x_1 k_1 + k_{12}(x_2 - x_1)$$

while the equation of motion for the second mass (with $m_2 = m$) is

$$m\ddot{x}_2 = -k_2 x_2 - k_{12}(x_2 - x_1)$$

. Now, assuming that $k_1 = k_2$ for simplicity, the two *coupled* equations which describe the motion of the springs are

$$m\ddot{x}_1 + (k + k_{12})x_1 - k_{12}x_2 = 0 \qquad (13.1)$$

$$m\ddot{x}_2 + (k + k_{12})x_2 - k_{12}x_1 = 0 \qquad (13.2)$$

For this special problem, these equations are easily solved. We will do it the easy way first and then develop the more general method after.

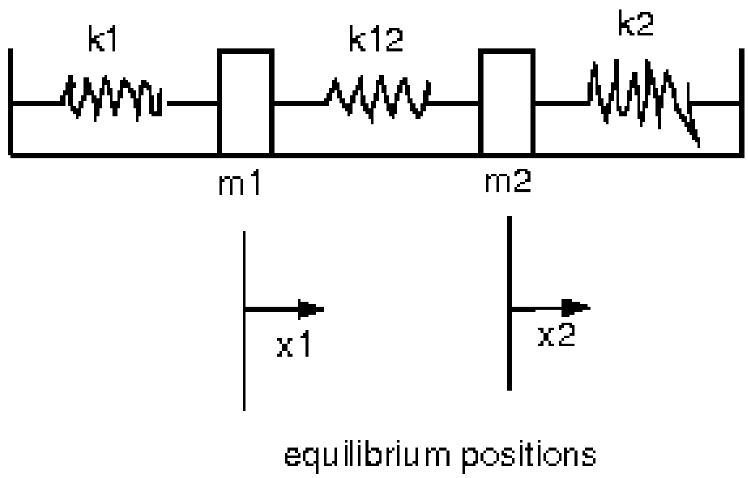


Figure 13.1: Coupled springs.

## 13.2 Easy solution

If we add equations 13.1 and 13.2 together, we get

$$m(\ddot{x}_1 + \ddot{x}_2) + (k + k_{12})(x_1 + x_2) - k_{12}(x_1 + x_2) = 0.$$

If, instead, we subtract the two equations, we get

$$m(\ddot{x}_1 - \ddot{x}_2) + (k + k_{12})(x_1 - x_2) + k_{12}(x_1 - x_2) = 0$$

Now, to make life easier, we can change to a new pair of variables,

$$\begin{aligned} \xi_1 &= x_1 + x_2 \\ \xi_2 &= x_1 - x_2, \end{aligned}$$

so that the new equations become

$$\begin{aligned} m\ddot{\xi}_1 + k\xi_1 &= 0 \\ m\ddot{\xi}_2 + (k + 2k_{12})\xi_2 &= 0. \end{aligned}$$

These equations are *uncoupled* and therefore the motions of $\xi_1$ and $\xi_2$ are independent. Both equations are those of a simple harmonic oscillator, so both motions are simple harmonic oscillation, the first with frequency

$$\omega_1 = \sqrt{k/m}$$

and the second with frequency

$$\omega_2 = \sqrt{(k + 2k_{12})/m}.$$

These frequencies are called the *characteristic frequencies* of the problem (you will also hear them called *eigenfrequencies*, which is just German for the same thing. But we won the war, so we will call them characteristic frequencies.)

The full solution to the problem is now

$$\begin{aligned} \xi_1 &= C_{11} \exp(i\omega_1 t) + C_{12} \exp(-i\omega_1 t) \\ \xi_2 &= C_{21} \exp(i\omega_2 t) + C_{22} \exp(-i\omega_2 t) \end{aligned}$$

where the $C_{ij}$ are complex constants of integration determined by the 4 required initial conditions (for example, the position and velocity of each of the 2 masses at time zero).

There are two normal (or characteristic) modes of oscillation that are associated with these frequencies. We can find them by cleverly choosing the right initial conditions. First, let's move both masses toward the center such that $x_1(0) = -x_2(0)$ and start the velocities in opposite directions such that $\dot{x}_1(0) = -\dot{x}_2(0)$, we then have that

$$\xi_1(0) = x_1(0) + x_2(0) = 0$$

and

$$\dot{\xi}_2(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0.$$

Then, since

$$\xi_1 = C_{11} \exp(i\omega_1 t) + C_{12} \exp(-i\omega_1 t)$$

it must be that $C_{11}$ and $C_{12}$ are zero and $\xi_1 = 0$ for all $t$, so at all times $x_1 = -x_2$ and $\dot{x}_1 = -\dot{x}_2$. The oscillation is in the anti-symmetrical mode shown in Figure 13.2, which has frequency $\omega_2 = \sqrt{(k + 2k_{12})/m}$

We can demonstrate the other characteristic mode by examining the solution when both masses are moving in the same direction, with initial conditions $x_1(0) = x_2(0)$, $\dot{x}_1(0) = \dot{x}_2(0)$. In this case, we get a similar constraint that $\xi_2(t) = 0$ for all $t$, so the only solution is

$$\xi_2(t) = C_{11} \exp(i\omega_1 t) + C_{12} \exp(-i\omega_1 t)$$

with $\omega_1 = \sqrt{k/m}$ which is the natural frequency of a single oscillator. This oscillation is in the symeetric mode illustrated in Figure 13.2. The frequency is the same as that of a single oscillator with a spring constant of one of the outer two springs which makes sense, because in this mode the spring is never stretched.

These two different modes of oscillation are called the *characteristic modes*, or the *eigenmodes* (argh!) of the oscillation. All oscillations that the system is capable of doing can be described in terms of non-interacting sums of these two modes. For coupled oscillators, the general mode of solution will be to identify these characteristics modes of oscillations and their characteristics frequencies.

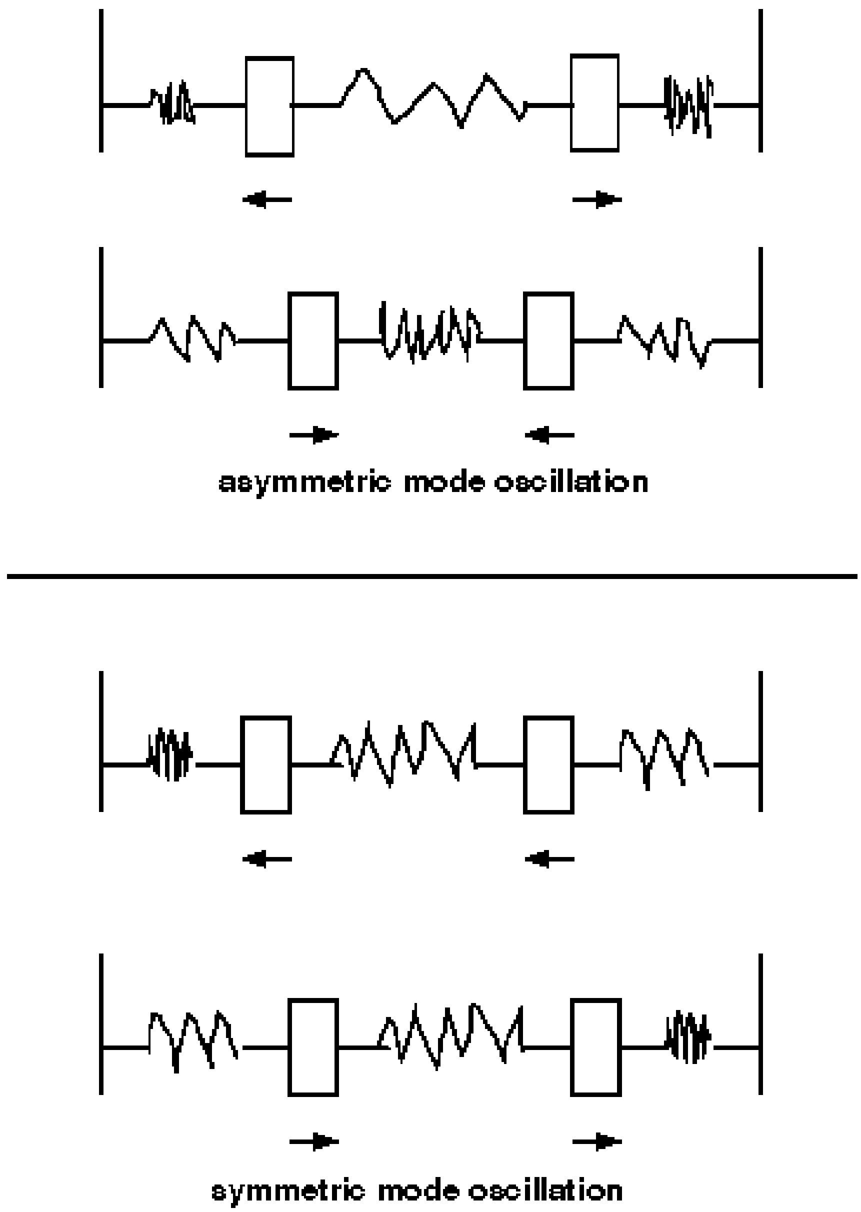


Figure 13.2: The two modes of the coupled spring solution. The spring can either oscillate in the antisymmetric mode where the two motions are equal but opposite, or it can oscillate in the symmetric mode where the motions are equal and in phase.

## 13.3 General solution

Most of the time, the coupled oscillator solution is not as easy to solve as the last one (in principle, they are *all* simple if you knew the right substitution to make, since they will all consist of several characteristic modes, but finding what these is is not always simple.) We will, therefore, solve the same equation as before, but using the general method. Rewriting the coupled differential equations:

$$\begin{aligned} m\ddot{x}_1 + (k + k_{12})x_1 - k_{12}x_2 &= 0 \\ m\ddot{x}_2 + (k + k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned}$$

Following our usual practice, we can guess that some sort of harmonic function is a solution, so that

$$\begin{aligned} x_1(t) &= B_1 \exp(i\omega t) \\ x_2(t) &= B_2 \exp(i\omega t) \end{aligned}$$

where, in general, the $B$s are complex numbers. If these indeed are solutions, what must the values of the $\omega$s be? Substituting in as usual, we find that

$$\begin{aligned} -m\omega^2 B_1 \exp(i\omega t) + (k + k_{12})B_1 \exp(i\omega t) - k_{12}B_2 \exp(i\omega t) &= 0 \\ -m\omega^2 B_2 \exp(i\omega t) + (k + k_{12})B_2 \exp(i\omega t) - k_{12}B_1 \exp(i\omega t) &= 0 \end{aligned}$$

which, after dividing through by the exponents, gives us the simultaneous characteristic equations

$$(k + k_{12} - m\omega^2)B_1 - k_{12}B_2 = 0 \quad (13.3)$$

$$-k_{12}B_1 + (k + k_{12} - m\omega^2)B_2 = 0 \quad (13.4)$$

Thing back to ancient times when you learned how to generally solve equations like this, for example, if you had

$$\begin{aligned} a_1 y_1 + a_2 y_2 &= c_1 \\ a_3 y_1 + a_4 y_2 &= c_2 \end{aligned}$$

the formal method for solving this is Cramer's rule, which states that

$$y_1 = \frac{\begin{vmatrix} c_1 & a_2 \\ c_2 & a_4 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}}$$

and

$$y_2 = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}}$$

In our problem, $c_1$ and $c_2$ are zero, so we have instead

$$\begin{aligned} y_1 &= \frac{\begin{vmatrix} 0 & a_2 \\ 0 & a_4 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}} \\ y_2 &= \frac{\begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}} \end{aligned}$$

So, in general, we are left which the solution that the $y$s are always zero. Of course, this is a solution to our problem: if nothing ever moves the equations are perfectly satisfied. The only way to have a non-zero solution to the problem is if the denominator is equal to zero and thus the solution becomes formally indeterminate. So we have the condition for a non-trivial solution be that

$$\begin{vmatrix} (k + k_{12} - m\omega^2) & -k_{12} \\ -k_1 2 & (k + k_1 2 - m\omega^2) \end{vmatrix} = 0$$

The determinate of this equation is

$$(k + k_{12} - m\omega^2)(k + k_{12} - m\omega^2) - (-k_{12})(-k_{12}) = 0$$

or

$$(k + k_{12} - m\omega^2)^2 - k_{12}^2 = 0.$$

This condition is the one that must be satisfied so that $x_1 = B_1 \exp(i\omega t)$ and $x_2 = B_2 \exp(i\omega t)$ will be solutions to the coupled equations. We solve this equation to find the characteristic frequencies of the problem. Thus,

$$(k + k_{12} - m\omega^2)^2 = k_{12}^2$$

so

$$(k + k_{12} - m\omega^2) = \pm k_{12}$$

or

$$m\omega^2 = k + k_1 2 \pm k_{12}$$

which shows that

$$\omega = \pm \frac{\sqrt{k + k_{12} \pm k_{12}}}{m}$$

and we see that there are 4 separate values of $\omega$ that work:

$$\begin{aligned} \omega_1 &= \pm\sqrt{k/m} \\ \omega_2 &= \pm \frac{\sqrt{k + 2k_{12}}}{m}. \end{aligned}$$

Note that these are the same frequencies that we derived above, only it took much more work to do it this way.

The general solution to the problem is a linear sum of the four independent solutions to the problem. That is, in this case, the real parts of

$$
\begin{aligned}
x_1(t) &= B'_{11}\exp(i\omega_1 t) + B''_{11}\exp(-i\omega_1 t) + B'_{21}\exp(i\omega_2 t) + B''_{21}\exp(-i\omega_2 t) \\
x_2(t) &= B'_{12}\exp(i\omega_1 t) + B''_{12}\exp(-i\omega_1 t) + B'_{22}\exp(i\omega_2 t) + B''_{22}\exp(-i\omega_2 t)
\end{aligned}
$$

where the $B_{ij}$s are complex constants to be found from the initial conditions, the subscript $i$ referes to the $i$th characteristic frequency, $\omega_i$, the subscript $j$ refers to the $j$th mass, and the primes refer to the positive and negative values of $\omega$.

Eight constants appear in the equation, but there can only be 4 independent constants associated with the 4 initial conditions, so many of the above constants must be related to each other. This fact is another way of saying that the motion of $x_1(t)$ is *coupled* tp $x_2(t)$ in general.

We can find these relationships from matrix equation 13.4 Let us first try the case for $\omega = \pm\omega_1 = \pm\sqrt{k/m}$. The equation becomes

$$
\begin{aligned}
(k + k_{12} - m(k/m))B_{11} - k_{12}B_{12} &= 0 \\
-k_{12}B_{11} + (k + k_{12} - m(k/m))B_{12} &= 0
\end{aligned}
$$

or simply

$$
\begin{aligned}
k_{12}B_{11} - k_{12}B_{12} &= 0 \\
-k_{12}B_{11} + k_{12}B_{12} &= 0
\end{aligned}
$$

and from *either* equation we see that

$$B_{11} = B_{12}$$

From the second characteristic frequency $\omega = \pm\sqrt{\frac{k+2k_{12}}{m}}$, we get

$$
\begin{aligned}
(k + k_{12} - m\frac{k+2k_{12}}{m})B_{21} - k_{12}B_{22} &= 0 \\
-k_{12}B_{21} + (k + k_{12} - m\frac{k+2k_{12}}{m})B_{22} &= 0
\end{aligned}
$$

so, again, for *either* solution we must have

$$B_{21} = -B_{22}$$

Now we can write the general solution in terms of the 4 independent constants to be found from the 4 inital conditions

$$
\begin{aligned}
x_1(t) &= B'_{11}\exp(i\omega_1 t) + B''_{11}\exp(-i\omega_1 t) + B'_{21}\exp(i\omega 2t) + B'_{21}\exp(-i\omega_2 t) \\
x_2(t) &= B'_{11}\exp(i\omega_1 t) + B''_{11}\exp(-i\omega_1 t) - B'_{21}\exp(i\omega 2t) - B'_{21}\exp(-i\omega_2 t)
\end{aligned}
$$

(Note: the 2nd subscript on the $B$'s is now superfluous; it was only used to help (?) clearify this development. Remember, the $B$s are all only constants of integration!)

The above solution should look somewhat familiar. If we re-introduce the characteristics coordinates

$$
\begin{aligned}
\xi_1 &= C_{11}\exp(i\omega_1 t) + C_{12}\exp(-i\omega_1 t) \\
\xi_2 &= C_{21}\exp(i\omega_2 t) + C_{22}\exp(-i\omega_2 t)
\end{aligned}
$$

where now the second subscript refers to the primes on the $B$s, we can see that $x_1(t)$ and $x_2(t)$ can be expressed as linear combinations of the characteristics coordinates and characteristic modes. Whew. Frequently, as in the next example, you will only want to know the characteristic frequencies (and once you do the modes themselves might be obvious, anyway) so you won't have to go through all of this mess. Instead, you will just have to set up the equations and solve from the characteristic frequencies. You will often hear this refered to as the eigen-(frequency, mode, value) method of determing the movements of these masses.

## 13.4 A molecular example

As a slightly different example, consider a $CO_2$ molecule and its linear vibrations, as illustrated in Figure 13.4. .

The equations of motion are

$$
\begin{aligned}
m\ddot{x}_1 &= k(x_2 - x_1) \\
M\ddot{x}_2 &= k(x_3 - x_2) - k(x_2 - x_1) \\
m\ddot{x}_3 &= -k(x_3 - x_2)
\end{aligned}
$$

or

$$
\begin{aligned}
m\ddot{x}_1 + kx_1 - kx_2 &= 0 \\
M\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 &= 0 \\
m\ddot{x}_3 + kx_3 - kx_2 &= 0
\end{aligned}
$$

which are the three coupled differential equations of the motion.

We expect harmonic motion, so we find the conditions for the various solutions of the form

$$
\begin{aligned}
x_1 &= B_1\exp(i\omega t) \\
x_2 &= B_2\exp(i\omega t) \\
x_3 &= B_3\exp(i\omega t)
\end{aligned}
$$

and we try to find the 3 eigenfrequencies for which these solutions work.

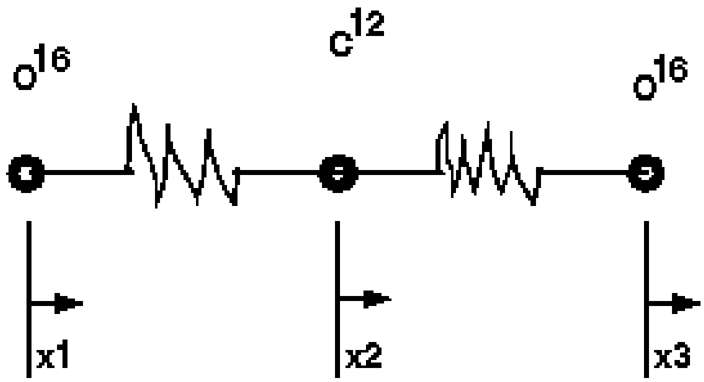


Figure 13.3: A $CO_2$ molecule

Putting these solutioins into the differential equations,

$$
\begin{aligned}
-\omega^2 m B_1 + kB_1 - kB_2 &= 0 \\
-\omega^2 M B_2 + 2kB_2 - kB_1 - kB_3 &= 0 \\
-\omega^2 m B_3 + kB_3 = kB_2 &= 0
\end{aligned}
$$

or, in matrix form,

$$
\begin{pmatrix} (k-\omega^2 m) & -k & 0 \\ -k & (2k-\omega^2 M) & -k \\ 0 & -k & (k-\omega^2 m) \end{pmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = 0.
$$

For non-trivial solutions, we need the determinant of the equation to be zero, thus

$$
\begin{aligned}
(k-\omega^2 m)[(2k-\omega^2 M)(k-\omega^2 m) - k^2] - (-k)[-k(k-\omega^2 m)] &= 0 \\
(k-\omega^2 m)[2k^2 - 2k\omega^2 m - \omega^2 Mk + \omega^4 Mn - k^2 - k^2] &= 0 \\
(k-\omega^2 m)[\omega^2(-2km - Mk + \omega^2 Mm)] &= 0 \\
(k-\omega^2 m)(\omega^2)(\omega^2 Mm - k(2m+M)) &= 0
\end{aligned}
$$

The three roots of this equation are

$$
\omega_1 = 0
$$

$$
\begin{aligned}
\omega_2 &= \pm\sqrt{k/m} \\
\omega_3 &= \pm\sqrt{\frac{k(2m+M)}{Mm}} = \pm\sqrt{\frac{k}{m}(1+\frac{2m}{M})}
\end{aligned}
$$

What are the characteristic modes of these three oscillations? We could go through the math and figure them out, but, instead, we will just assert that they must be those shown in Figure 13.4.

The first frequency corresponds to simple translation of the molecule, which hardly even counts. The second two, though, are two true fundamental vibrational modes of the molecule that emit photons. Let's compute the ratio of the two frequencies of these two modes for $CO_2$:

$$
\frac{\omega_3}{\omega_2} = \frac{\sqrt{\frac{k}{m}(1+\frac{2m}{M})}}{\sqrt{\frac{k}{m}}} = \sqrt{1+\frac{2m}{M}} = \sqrt{1+\frac{2(16)}{12}} = \sqrt{1+8/3} = 1.91
$$

Now what if, instead, we look at the isotopes of $CO_2$? For $C^{13}O_2^{16}$, we get a ratio of $\sqrt{1+32/16} = 1.732$, while for $C^{12}O_2^{18}$ we get $\sqrt{1+36/12} = 2$.

Thus we can see that the frequencies, and thus the emission and absorption frequencies of these lines shift around with isotpoic substitution. This fact allows one, for example to try to measure isotopic compositions of atmospheric gases *in situ* with a small absorption spectrometer looking at the different wavelengths corresponding to the different isotopes. In real life, these things are much more complicated, as the quantum effects of the vibrations and the addition of rotational splitting of the lines causes significantly more complicated spectra.

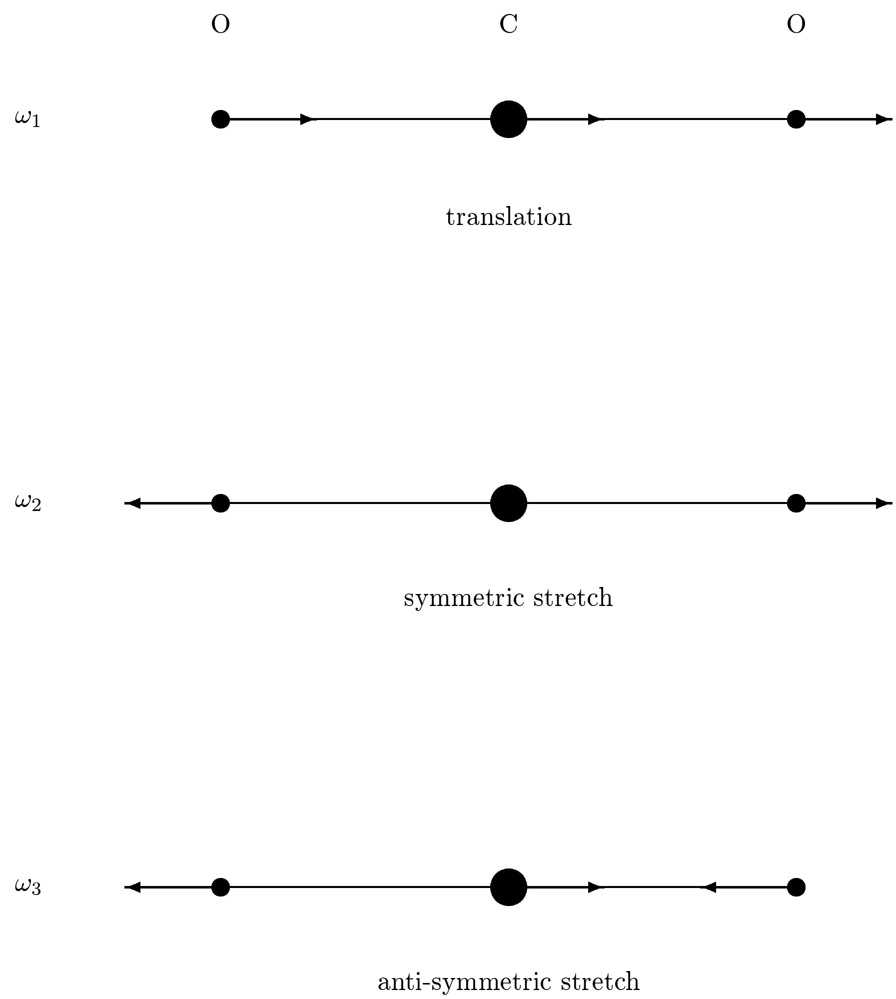


Figure 13.4: Modes of $CO_2$ vibration