Lecture 11

Oscillations

11.1 Simple harmonic oscillators

A dynamic system which is simple yet important for the earth sciences is that of the *simple harmonic oscillator* (SHO), which consists of a single mass oscillating in the absence of external forcing or internal damping. Three such simple systems, illustrated in Figure 11.1, are a mass connected to a spring moving on a frictionless plane, as mass in a vacuum bouncing on the end of a spring, and a frictionless pendulum in a vacuum. In all cases we have to specify “frictionless” or “in a vacuum” to prevent *damping* of the oscillation, which will change the simple form of the equations. These systems all lead to the same second order linear differential equation, the harmonic oscillator equation. This equation is the basis for a great range of problems in physcis, from quantum mechanics, electromagnetic waves, oscillating crystals, to ocean waves.

We'll first consider the case illustrated in Figure 11.1(a): a mass attached to an ideal spring sliding on a frictionless plane. First off, what do we mean by an ideal spring? This is a spring that obeys perfectly *Hooke's Law*, that the force exerted by the spring is directly proportional to the stretching length of the spring:

$$F = -k\Delta x,$$

where k is called the spring constant, with units of force per length, and Δx is the amount that the spring is stretched. The force is negative because the spring always opposes the displacement of the mass away from the center.

Just like the case of gravitational potential, we can define potential energy stored in the spring. We could go through the same sort of calculation, figuring the potential energy by determining the kinetic energy given to a mass, or we could go to the more fundamental definition of the potential that we found, namely that the potential is the work required to move an object from one

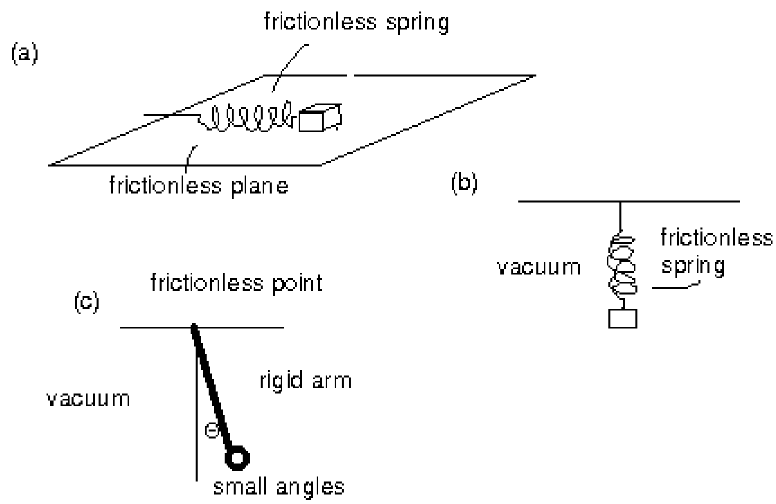

Figure 11.1: Three examples of simple harmonic oscillators.

position to another, or

$$U = -\int_0^{x_1} F\ dx$$

For the spring equation, the potential becomes

$$U(x) = \int_0^x kx'\ dx'$$

or

$$U(x) = \frac{k}{2}x^2$$

where we have choosen the constant of integration such that the spring has zero potential when it is uncompressed, which makes sense.

Using the principle of conservation of energy, we can now easily determine the equations controlling this simple harmonic oscillator. Because potential + kinetic energy must equal a constant,

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = C$$

or, rewriting as a differential equation,

$$\frac{dx}{dt}^2 + \frac{k}{m}x^2 = C.$$

Non-linear differential equations are a pain, even if they *are* first order. We can turn this one into a *second*-order linear DE by simply taking the derivative with respect to x on both sides:

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{k}{m}x\frac{dx}{dt} = 0$$

or

$$\frac{dx^2}{dt^2} = -\frac{k}{m}x,$$

which is the classic simple harmonic oscillator equation.

This equation is a second order version of the population, and, like that equation, has an obvious exponential solution:

$$x = A\exp(\omega t),$$

where, for this solution to work, it must be that $\omega^2 = -k/m$ or $\omega = i\sqrt{k/m}$, so the true solution must be

$$x = A\exp(i\sqrt{\frac{k}{m}}t).$$

Recall, though, that for a second-order differential equation there must be two independent solutions, and that any combination of these two solutions must also be a solution. The second solution is quite obviously

$$x = B\exp(-i\sqrt{\frac{k}{m}}t)$$

so the complete general solution is

$$x = A\exp(-i\sqrt{\frac{k}{m}}t) + B\exp(i\sqrt{\frac{k}{m}}t).$$

The constants A and B are given by the initial conditions, where now there must be 2, because we have a second order DE. Two such initial conditions that would obviously suffice to describe the motion for all susequent times would be the position and velocity of the mass at any time t.

Just to show that this equation pops up in many places, let us next consider the case of a pendulum swinging on a rigid rod of lenght l. At first look, the physics seems very diffenent (Figure 11.1). The potential of the pendulum is the integral of the force over the distance, or

$$U(\theta) = \int_0^{\theta_1} mg\sin\theta d\theta$$

or

$$U(\theta) = -mgl\cos\theta + C$$

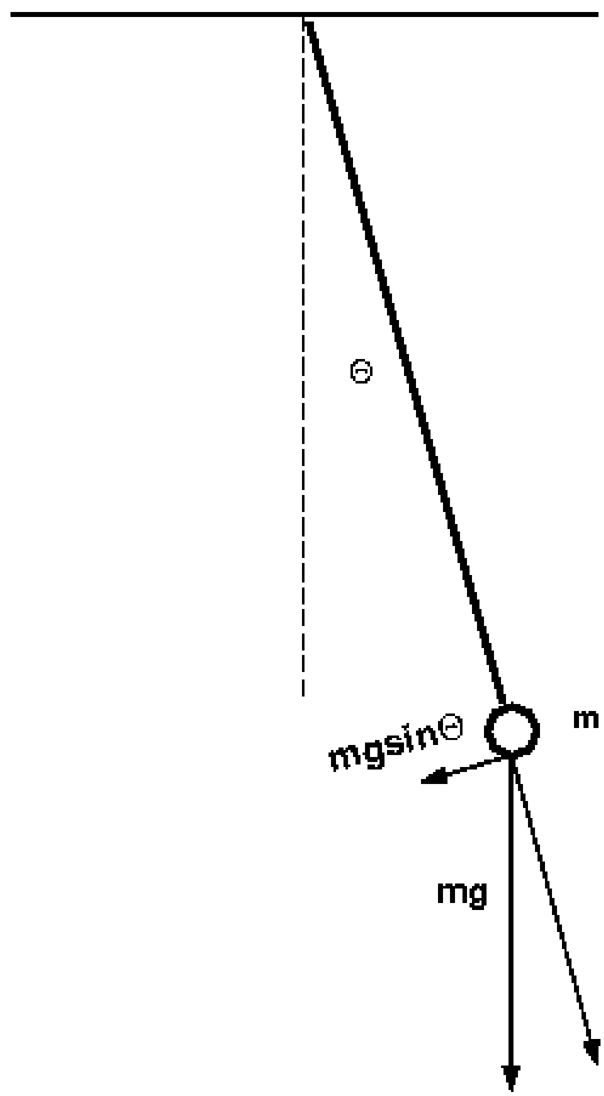

Figure 11.2: Geometry of the perfect pendulum.

where we will chose C to be zero, since it's arbitrary.

The kinetic energy is

$$K = \frac{1}{2}ml^2(\frac{d\theta}{dt})^2,$$

so the energy balance equation becomes

$$\frac{1}{2}ml^2(\frac{d\theta}{dt})^2 - mgl\cos\theta = C$$

which we again differentiate to obtain

$$\frac{d^2\theta}{dt^2}ml\frac{d\theta}{dt} + \frac{d\theta}{dt}mg\sin\theta = 0$$

which becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

which *isn't* the same equation that we had above!

But if we make the requirement that $\theta \ll 1$, the equations become the same. This is because we can make a Taylor expansion of $\sin\theta$ recalling that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + ...$$

which, for $f(x) = \sin(x)$ and $x \ll 1$ becomes

$$\sin(x) \equiv x.$$

So our pendulum equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

which is now just the SHO equation with $k/m = g/l$! If we allow θ to get large, though, higher order terms begin to creep in, and the equation is no longer solvable.

11.2 Full solution

With the general solution being

$$x = A\exp(i\sqrt{k/m}t) + B\exp(-i\sqrt{k/m}t)$$

we now try for a specific solution for some initial conditions.

Consider a mass m attached to spring with spring constant k which is initially pulled to a displacement x_0 away from the center and then let go. The initial velocity, then, is zero. We plug the initial conditions into the solution:

$$x(t = 0) = A + B = x_0$$

and

$$v(t=0) = \frac{dx}{dt} = i\sqrt{k/m}(A-B) = 0$$

So the solution is $A = B = x_0/2$, and the real part of the total solution becomes

$$x(t) = x_0 \cos(\sqrt{k/m}t)$$

where now $\sqrt{k/m}$ is ω, the *angular frequency* of the oscillation (simply related to f, the frequency, by $f = \omega/2\pi$). Let's examine the effects of different parameters. If we replace the mass with something heavier, the oscillation frequency decrease: the oscillation gets slower, which makes intuitive sense. Alternatively, if we replace the spring with a stiffer one (larger spring constant), the oscillation frequency goes up, which also makes intuitive sense.

11.3 Phase

An alternative to searching for solutions to the values of A and B is to assume a solution of the form

$$x(t) = C\exp[i(\omega t + \phi)]$$

where now C and ϕ are our two constants of integration. That these two are equivalent can be seen from explicitly expanding the formula:

$$\begin{aligned}
x(t) &= C\exp[i(\omega t + \phi)] \\
&= C[\cos(\omega t + \phi) + i\sin(\omega t + \phi)] \\
&= C[\cos\omega t\cos\phi - \sin\omega t\sin\phi + i\cos\omega t\sin\phi + i\sin\omega t\cos\phi] \\
&= C[\cos\phi(\cos\omega t + i\sin\omega t) + \sin\phi(-\sin\omega t + i\cos\omega t) \\
&= C[\cos\phi(\cos\omega t + i\sin\omega t) + \sin\phi(\cos(-\omega t + \frac{3\pi}{2}) + i\sin(-\omega t + \frac{3\pi}{2}))] \\
&= C[cos\phi\exp(i\omega t) + \sin\phi\exp(-i\omega t + \frac{3\pi}{2})] \\
&= A\exp(i\omega t) + B\exp(-i\omega t)
\end{aligned}$$

This new form is much more convenient. If we take the real part of the solution,

$$x(t) = C\cos(\omega t + \phi)$$

we see that the solution is a simply cosine oscillation, but with a phase that depends on ϕ. Matching initial conditions is now simple, particularly if we intuitively know where in the phase the initial conditions refer to. For example, what if the initial conditions are that $x(0) = 0$ but that $v(0) = v_0$? We can immedaitely see that the phase of the solution must be a place where the cosine function is zero but the maximum positive derivative, which means the phase must be $3\pi/2$. Simple substitution then shows that C must be v_0/ω.

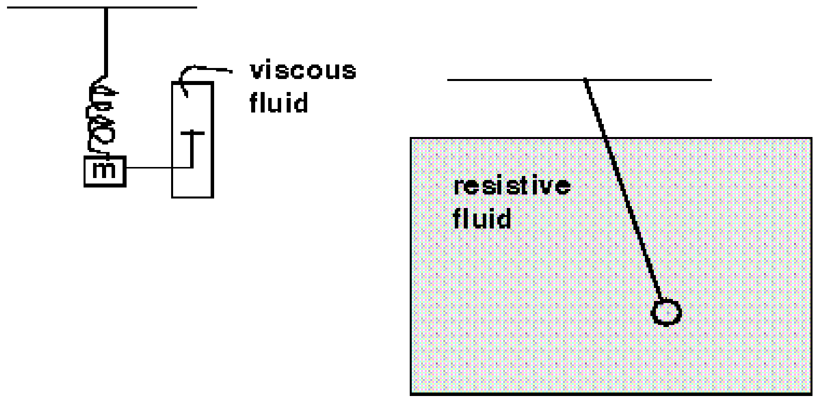

Figure 11.3: A damped oscillator.

11.4 Damped Oscillators

In real life, of course, nothing is simple, and the same holds for harmonic oscillators. The most common thing that will happen to an oscillator is that it will be damped, either by the resistive force of air on the pendulum, the friction of the plane, or the viscosity of the shock absorber fluid. Because of these actions, oscillations do not continue forever. The most generic cases of damped oscillations are those shown in Figure 11.4. In these examples, the damping force is proportional to velocity, if the velocity is slow. This is called the *linear approximation* to damping. For larger velocities, higher order terms need be considered, but for this simple linear term, a linear DE again results. Assuming force balance, we should be able to write an equation where the spring force is equal to the inertial force plus the resitive force, or

$$-kx = m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt},$$

where γ is called the damping "constant" which, in general, may not be a constant. But if it is indeed a constant, we get the second order DE

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

which is analytically solvable. If γ (or k or m) vary with time or with velocity we have to resort to numeric methods to solve the equation.

How do we generally solve such an equation with γ (and everything else) equal to a constant? Sad to say, such a general discussion is beyond the scope of this class (I've always wanted to say something like this...), but we will assert that the solution should be something like an exponentially damped sinusoidal oscillation, or, in general, the solution should look something like

$$x = A\exp[i(ct + \phi)]$$

By substitution, we then get

$$-mc^2 + i\gamma c + k = 0$$

where now it becomes clear that c is, in general, a complex number that we can define as

$$c = c_1 + ic_2,$$

where c_1 and c_2 are real. For a complex number to equal zero, both its real and imaginary parts must separately equal zero, so if we substitute the imaginary form of c back into the equation and split out the real and imaginary parts, we get two equations:

$$\begin{aligned} -mc_1^2 + mc_2^2 - \gamma c_2 + k &= 0 \\ -2mc_1c_2 + \gamma c_1 &= 0 \end{aligned}$$

From the second of these eqautions we get that

$$c_2 = \frac{1}{2}\frac{\gamma}{m}$$

which we can now substitute back into the first equation to get

$$c_1^2 - \frac{1}{4}\frac{\gamma^2}{m^2} + \frac{1}{2}\frac{\gamma^2}{m^2} - \frac{k}{m} = 0$$

which just becomes

$$c_1 = \pm\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

Thus if $c = c_1 + c_2$ then $x = A\exp(i(\alpha t + \phi))$ is a solution to the differential equation. To see what's going on, we should really take the real part of this equation, thus

$$\begin{aligned} x &= \text{Real}\left\{A\exp[i(ct + \phi)]\right\} \\ &= \text{Real}\left\{A\exp[ic_1t - c_2t + i\phi]\right\} \\ &= \text{Real}\left\{A\exp[it\sqrt{\frac{k}{m} - \gamma^2\ 4m^2} - \frac{1}{2}\frac{\gamma}{m}t + i\phi]\right\} \\ &= A\exp(\frac{-\gamma}{2m}t)\cos(t\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} + \phi) \end{aligned}$$

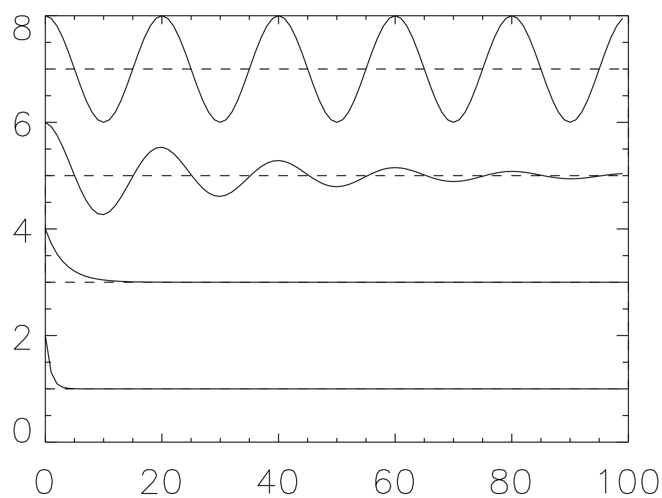

Figure 11.4: Examples of oscillations in an undamped, a lightly damped, a critically damped, and an over-damped system.

This solution is one of an exponentially damped oscillation. For the case of no damping $\gamma = 0$, the SHO solution comes back out, and the oscillation frequency is the natural frequency, $\omega = \sqrt{k/m}$. When damping is added, not only does the magnitude of the oscillation decay (with an e-folding time of $2m/\gamma$, so a larger γ means a faster decay, which makes sense), but the frequency is also decreased and the oscillations occur more slowly, which also makes sense. The new angular frequency is

$$\omega = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

But what happens if the quantity in the square root is less than zero, that is $\gamma^2/4m^2 > k/m$? In this case ω is imaginary and the solution becomes one of pure exponential decay with no oscillation. The system is said to be *over-damped*. If $\omega = 0$ the system is said to be *critically-damped*, and, again, only decay occurs. Figure 11.4 shows cases of oscillations with no damping, small damping, critical damping, and over damping.