Lecture 10

Gravitational potential

10.1 Uniform field

Potential energy is, just as it sounds, energy that is stored and available for future use. The energy can be stored in the form of a tightly wound spring, a chemical bond, or a rock on a cliff poised to fall. This later form of potential energy is gravitational potential. Gravitational potential energy is most easily turned into kinetic energy; the rock falls from the cliff and gains speed and thus energy as it plummets to the ground. We can determine how much potential energy the rock had at the top of the cliff by figuring out how much kinetic energy it has at different points in its flight. Consider a cliff of height h from which the rock falls. From basic ballistic physics we recall that

$$v = gt$$

and

$$s = v_0 t + \frac{1}{2}gt^2$$

(the second equation, of course, comes from just integrating the first with $ds/dt = v$ and the initial condition that $v(t = 0) = v_0$). With a zero initial velocity, the time the rock takes to fall to the bottom of the cliff is

$$t = \sqrt{2h/g}$$

so the velocity at the bottom of the cliff is

$$v = \sqrt{2hg}.$$

Kinetic energy is defined as

$$K = \frac{1}{2}mv^2,$$

so the kinetic energy of the rock at the bottom of the cliff is

$$K = mgh.$$

We will take this equation as the definition of gravitational potential.

Just before the rock hits the ground at the bottom of the cliff all of its potential energy has been converted into kinetic energy and its potential energy is zero (what happens to all of this energy when the rock *hits* the ground?). But now what if we take the same rock, roll it along the ground a little until we find a new cliff and push it off. Suddenly the rock that we just said had zero potential energy has new potential all over again and is busily converting it into kinetic energy. So without knowing what cliffs are ahead, how do we ever know how much potential energy a rock *really* has? The answer is that it does not matter. Potential energy is only defined as a relative number, so we can only speak of potential energy between two points in space. So for the above example, the potential energy of the rock is really defined only as the potential of the rock above the bottom of the cliff (or we could define it as the potential of the rock above the second cliff, or whatever else we like).

As illustrated in Figure 10.1, a ball rolling along a bumpy road will convert potential energy of high points to kinetic energy at low points. The velocity at the low point does not depend on overall elevation of the road, only on the relative elevation difference between the high and low points of the road.

With these definitions of kinetic and potential energy and our knowledge of conservation of energy, we can quickly calculate many simple problems in motion. For example, we are on a sled on a hill with a 10% grade (which is pretty steep). How fast are we going when we hit the tree at the bottom of the hill, which is 100 m lower than the top of the hill (and thus the whole hill is slightly more than 1 km long). First, we do out the answer explicitly, as illustrated in Figure 10.1. The acceleration felt by the sled is equal to the vector component of gravity in the direction of the sled motion, which is $g \sin\theta$ where $\theta = \tan^{-1} 0.1$, because we are on a 10% grade. Thus $\sin\theta = \frac{100}{\sqrt{100^2+1000^2}} = 0.0995$ (θ is about 6 degrees), so the acceleration is 9.8×0.0995 m/s$^2 = 0.975$ m/sec^2. The distance that the sled has to travel is $\sqrt{100^2 + 1000^2} = 1005$ m. From the formulae above, the time it takes to go this distance is $t = \sqrt{2h/a} = 45$ seconds, and the velocity is $v = 45 \times 0.975 = 44$ m/s (which is just about 100 miles/hour).

But what a waste of time this calculation was! Instead, we know that the potential energy at the top of the hill was converted into kinetic energy at the bottom, so that

$$\frac{1}{2}mv^2 = mgh$$

or

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 100} = 44 m/s.$$

Nowhere in this calculation did we even bother with slopes or anything else. The velocity is the same at the bottom of the hill no matter what the slope is.

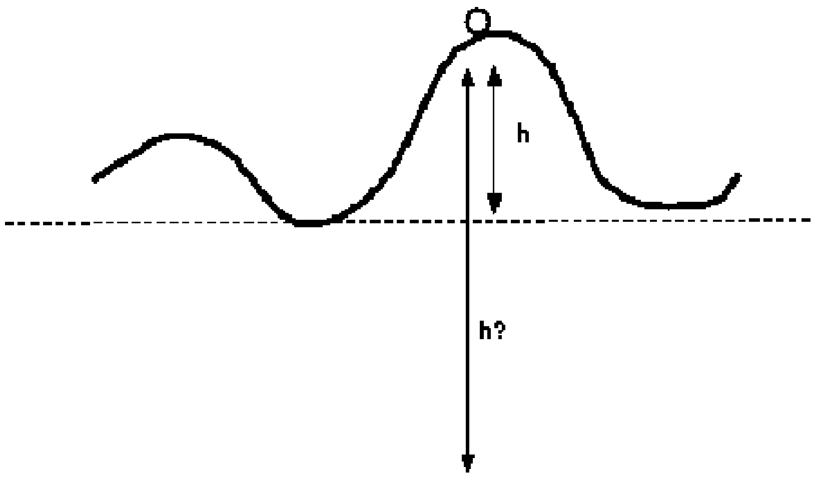

Figure 10.1: Illustration of potential energy in a uniform gravitational field. The potential energy of the ball at the top of the hill relative to the valley does not change if the elevatation of the entire configuration changes.

For steep slopes, the *time* required to get to the bottom is shorter, but the final velocity depends only on the total hill drop. To make this abundantly clear, let's redo the calculation for a 45% grade (which is really really really steep). The acceleration felt by the sled is $g \sin\theta = 6.9$ m/s^2, and the distance to be traveled is $\sqrt{100^2 + 100^2} = 141m$. The time it takes to travel this distance is $t = \sqrt{2 \times 141/6.9} = 6.4$ seconds (fast!) and the velocity upon tree impact is $v = 6.4 \times 6.9 = 44$ m/s! So it does not matter if you slide down a slope or jump off a cliff, when you hit the bottom you are always going the same speed, because all of your potential energy has been converted into kinetic energy. (In real life, of course, some of your potential energy is released through friction, so you might choose your option that gives the most friction. It is hard to do much work from friction if you jump off of a cliff, but if you're sliding friction is easier to come by.)

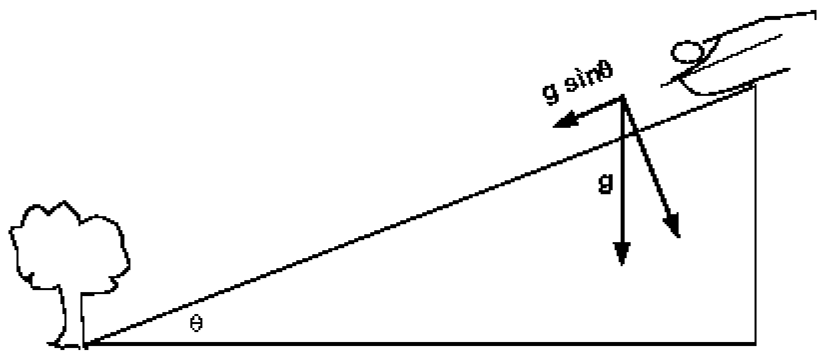

Figure 10.2: Sledding toward disaster. The acceleration of the sled is equal to the vector component of gravity in the direction of the sled motion.

10.2 Generalized potentials

If we now allow the gravitational field to vary with distance (as it really does), the potential becomes more complicated. We can no longer simply say that the potential is directly proportional to the distance between objects, because the gravitational field changes with distance. We can use the same method as before to find the potential energy, though. Consider a rock with mass m and initial velocity zero at a distance r from the sun (mass M). The acceleration of the rock due to the gravitational pull of the sun will be

$$a = -G\frac{M}{r^2},$$

and it will vary as the rock gets closer.

We can write the general differential equations:

$$\begin{aligned} \frac{dr}{dt} &= v \\ \frac{dv}{dt} &= -G\frac{M}{r^2} \end{aligned}$$

Integrating the second gives

$$v = -G\int \frac{M}{r(t)^2}dt$$

These are not integratable in their present form because we don't know $r(t)$ (and if we did that would be the answer we were looking for anyway!). Let's substitute variables into the second equation by changing $r(v)$ for $r(t)$ and using $dr = vdt$ to get

$$v = -G\int \frac{M}{v(r)r^2}dr$$

and now differentiating with respect to r to get

$$\frac{dv}{dr} = -\frac{GM}{vr^2}.$$

The variables are separable in this first-order differential equation, so we can immediately integrate this to be

$$\int vdv = -\int \frac{GM}{r^2}dr$$

or

$$\frac{1}{2}v^2 = \frac{GM}{r} + C,$$

where C is evidently the initial condition which relates to the initial velocity which we defined to be zero and to the initial potential energy which we know to be arbitrary. And now we have the answer, because the kinetic energy is $K = \frac{1}{2}mv^2$, so the potential energy that the rock has released is some constant plus the new kinetic energy, so it is

$$V = C - \frac{GMm}{r}$$

By convention, potential energies are defined so that two objects at infinity have zero potential energy with respect to each other, so objects closer always have *negative* potential energy (again, it is all relative and we only care about changes in potential energy anway), so

$$V = -\frac{GMm}{r}$$

And, finally, we will define *gravitational potential*, as opposed to potential energy, as a function independent of the mass of the rock, so

$$U = -\frac{GM}{r}$$

As usual, we could also define the potential from a collection of objects as the sum of the individual potentials, and the potential from a continuum of objects as the integral over the density function of the contiuum.

An easier way to have defined potential is as the energy required to move an object from one spot to another. Energy is defined as force times a distance, or for a variable force,

$$E = \int F\, ds,$$

so we could say that the potential difference between points r_1 and r_2 in the presence of the sun is

$$E = \int_{r_1}^{r_2} \frac{-GMm}{r^2} dr$$

and we get the same answer as above. But this form tells us something interesting, that the gravitational field is the differential of the potential. First, let's see how this works back in the case of the uniform field of the earth. Here we said that that the potentail energy is $V = mgh$, so the mass-independent potential $U = gh$ and $dU/dh = a = g$, which is what we know already. But wait, there's more. What if we try to differentiate in the horizonal direction instead of the vertical direction. Then the differnece in potential is zero so likewise the force derived is zero, like we again already knew. This fact tells us somthing important: the differentiation needs to be done separately in each spatial direction to get the force in each spatial direction. In three dimensions, this is written as

$$a = \nabla U,$$

where ∇ is a vector defined as $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, which means that to find the force in the x direction, take the partial derivative with respect to x. In full vector format,

$$\mathbf{a}(x, y, z) = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z})$$

where now $\mathbf{a}$ is explicitly a vector.

10.3 Using potentials

Just like potentials on the earth were useful for letting you easily calculate velocities and energies of objects, so are these full gravitational potentials. For example, what if we wanted to know how fast we have to leave the surface of the earth if we want to make it to the moon? The gravitational potential difference between the surface of the earth and the moon is (ignoring the mass of the moon)

$$U = -G\left(\frac{M_{\text{earth}}}{r_{\text{earth}}} - \frac{M_{\text{earth}}}{r_{\text{tomoon}}}\right)$$

where r_{tomoon} is the distance *to* the moon. Throwing in our favorite values ($G = 6.7 \times 10^{-11}$ J m^2 kg^{-2} , $M_{\text{earth}} = 6 \times 10^{24}$ kg, $r_{\text{earth}} = 6.4 \times 10^6$ m, and $r_{\text{moon}} = 3.8 \times 10^8$ km) we get a value of 6.2×10^7 J/kg. This is the amount of kinetic energy that we must have leaving the earth to make it to the moon. The velocity is therefore 11 km/s, which is pretty fast. Rockets only make it to the moon because they expend fuel along the way to give them an extra boost (converting *chemical* energy into kinetic energy). What if instead of stopping at the moon, we wanted to escape the earth forever? The velocity required to do this is aptly called the *escape velocity.* If we had enough velocity that the gravitational potential between the surface of the earth and a point infinitely far away were exactly equal to the kinetic energy we would have *exactly* the escape velocity. This means that when we got infinitely far from the earth we would have zero velocity. If we want to be moving as we get far from the earth, we need to have even more velocity. The escape velocity is easily seen to be

$$v_{\text{escape}} = \sqrt{\frac{2GM_{\text{earth}}}{r_{\text{earth}}}} = 11\text{km/s}.$$

To this degree of accuracy, this is the same as the speed required to reach the moon. Most of the work is evidently done very near the surface of the earth!

10.4 The Geoid

The earth is, of course, not round. It bluges at the equator due to the effects of rotation. This effect is easily understood in terms of gravitational potentials. To first approximation, let's consider the earth to be a perfectly fluid body. In this case there can be no potential energy differences between any points on the surface because, just like in the oceans, if there were a bump or a divot water would rush out or in to make the surface flat again. The rotation of the earth causes a *centrifugal force*, which can be thought of as an extra potential that we need to add to the gravitational potential. We all know that the centrifugal force, F_{c}, can be written as

$$F_{\text{c}} = \frac{mv^2}{r} = m\omega^2 r,$$

where ω is the rotation frequency of the planet. We know from above that $a = dU/dr$, so we can integrate to find the centrifugal potential,

$$U_{\text{c}} = \frac{1}{2}\omega^2 r$$

where we could have also added a constant of integration, but because potential are arbitrary to a constant anyway we will not.

Now we will require that the gravitational potential of the fluid at the pole be equal to the gravitational plus centrifugal potential of the fluid at the equator:

$$U_{\text{pole}} = \frac{-GM}{r_{\text{pole}}} \quad \text{and}$$

$$U_{\text{equator}} = \frac{-GM}{r_{\text{equator}}} + \frac{1}{2}\omega^2 r_{\text{equator}}$$

so

$$\frac{-GM}{r_{\text{pole}}} = \frac{-GM}{r_{\text{equator}}} + \frac{1}{2}\omega^2 r_{\text{equator}}$$

which reduces to

$$r_{\text{equator}} = r_{\text{pole}}[1 + \frac{\omega^2 r^3_{\text{equator}}}{2GM}].$$

Now we could either solve the quadratic equation for the new radius or we could cheat and say that we know that the equatorial and polar radii are going to be very close to each other, so we can just substitue in the right side of the equation and approximate

$$r_{\text{equator}} = r_{\text{pole}}[1 + \frac{r^3_{\text{pole}}\omega^2}{2GM}]$$

For the terrestrial parameters listed above, we calculate that the equatorial radius is 0.17% greater than the polar radius, or about 11 km.

Our calculated value is only about half of the real value of the equatorial bulge, which is about 21 km. What did we do wrong? If you think carefully you'll see that we cheated in one spot. We used the fact that the earth's gravitational field could be considered to be from a point mass at the center of the earth, but this fact only works if the earth is spherically symmetric, which we now know is not exactly true! The equatorial bulge causes a little bit of extra gravitational pull at the equator, which means that the potential is lower so that fluid can flow from the poles to the equator. This flow continues until the potential is the same everywhere.

Though our calculation is slightly off, we will use it nonetheless to approximation the complete shape of the earth as a function of latitude, θ. We need only modify our equation slightly to account for the fact that the centrifugal potential of particles changes with latitude because the velocity becomes $r\omega\cos\theta$ (note the limiting cases for $\theta = 0$ and $\theta = 90$ degrees). We can then simply rewrite the radius as

$$r(\theta) = r_{\text{pole}}[1 + \frac{r^3_{\text{pole}}\omega^2\cos^2\theta}{2GM}]$$

This equation states that the earth should be some sort of ellipsoid. Figure 10.4 shows the shape that we calculated, with the equatorial bulge exaggerated by a factor of 100.

This nice approximation to the earth (with the appropriate corrections we didn't make) is sometimes called the *reference ellipsoid*. Calculated even more accurately, to take into account the non-uniformities of the surface of the earth, this would be called the *geoid*. At first it was assumed that the geoid was something very much like a nice uniform ellipsoid like we calculated, but when

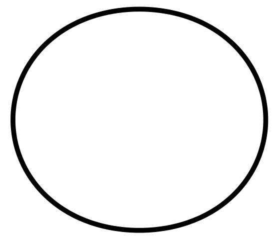

Figure 10.3: The calculated shape of the earth with the equatorial bulge exaggerated by a factor of 100. The real earth has a bulge almost twice as large as the one calculated.

the first US satellite went up in 1959 it was observed that the orbit that the satellite followed did not agree precisely with the calculations based on a near-ellipsoidal geoid. It was concluded that the geoid is best approximated, not by an ellipsoid, but by a slightly pear-shaped figure, with the small end of the pear being in the northern hemisphere and extending about 15 m above the reference ellipsoid.