Lecture 4

Applications of diffusion

4.1 Cooling of the earth

Last lecture, we discussed the error function as a functional solution to the diffusion equation and discussed its general properties that made it useful, namely that $\mathrm{erf}(0) = 0$ and that $\lim_{x\to 0} \mathrm{erf}(x) = 1$. Remember that the way that we are usually going to "solve" the diffusion equation is simply by trying to find one of the known functional forms (or sum of functions, since we know that sums of solutions are solutions). So anytime we have initial conditions that are matched by some sort of constant along a boundry plane, we can try to match it with an error function. As the best example of this type of solution, let's consider the cooling of the earth in the absence of any internal heat sources.

We can make a simplistic model of the earth by assuming it is initially at a temperature of 2000 K (the initial heat is from the initial gravitational collapse: potential energy has been converted to heat energy), but that the surface is in equilibrium with the solar radiation and stays at a temperature of 300 K, so the earth cools. What is the temperature with depth as a function of time? We simply want a functional solution to the diffusion equation where one boundry stays fixed and everything else is allowed to wander. Consider then the solution

$$T(z,t) = T_s + T_\infty \mathrm{erf}(z/2\sqrt{k}t)$$

where T_s is the 300 K surface temperature and T_∞ is the initial tempoerature everywhere else (actually it is the initial temperature minus the surface temperature).

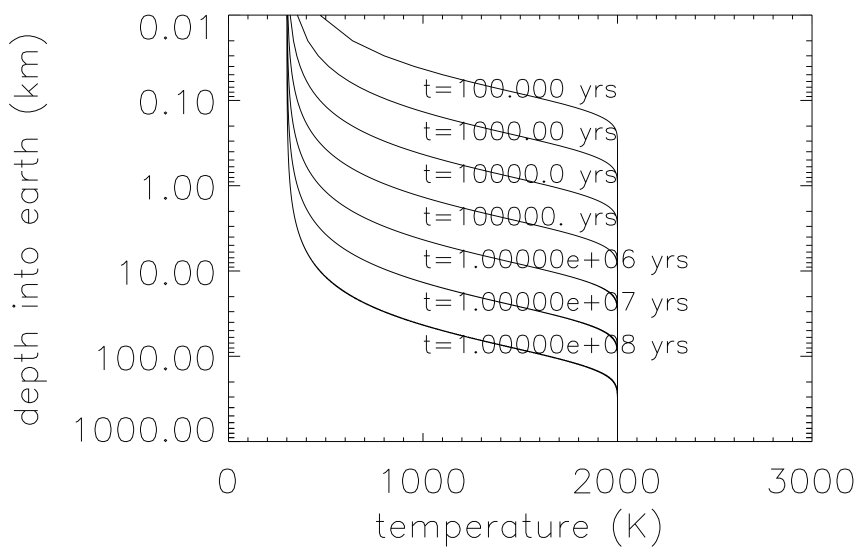

Figure 4.1: The temperature of the earth as a a function of depth and time, assuming no internal sources and a fully granite earth.

Let's make sure this fits everything we have as initial conditions. As $t = 0$, the value of the error function goes to 1 for every z except for $z = 0$, where the value goes to 0. Thus this function fits the initial condition, and we know it's a solution of the diffusion equation, so it must be the answer.

Assuming that the whole of the earth is granite, with $k = 10^{-6}$ m^2/s, we can plot the solution, as seen in Figure 4.1. The whole earth gradually cools to the temperature of the surface.

Lord Kelvin was the first to try to use this solution to come up with the (incorrect) age of the earth. How did he do it? The most mathematically obvious way would be to drill several thousand kilometers down and try to determine where the temperature plateaus. This doens't work in practice, though, but from the earliest days of mining it was known that a "geothermal gradient" of about 10 to 50 C per kilometer existed. Figure 4.1 shows the gradient found in our solution to the diffusion equation (in practice, what is really plotted is the difference in temperature at the surface and at 1 km depth). You can see that the earth's thermal gradient is reached after

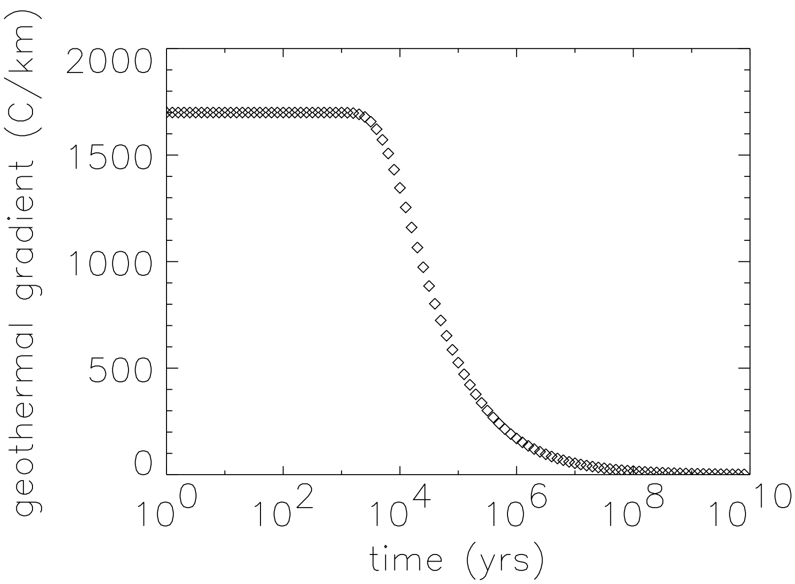

Figure 4.2: The thermal gradient in the first kilometer for the cooling earth as a function of time.

something like 10^7 years, thus that is the age of the earth, says Lord Kelvin! (Lord Kelvin really said 94×10^6 years to get down to 10 C/km, which is remarkably close to what we calculated). What did he do wrong? The standard answer is that, of course, the earth is heated from within by the radioactive decay of elements. But other things that change the answer alot that he didn't consider are chemical changes, phase changes, and the increase in melting point with pressure. (The problem was solved corrected in 1940 by Jeffreys who got a cooling time of about 2 billion years).

4.1.1 Separating the function

A general class of solutions that will lead to an interesting particular solution can be had by a nice general method called separating the function. Since we are merely looking now for any solution to the diffusion equation, why not consider one of the form

$$T(x,t) = g(x)f(t) + c,$$

where g and f are completely independent functions and c is just a constant (which we can easily see from the equation will always be able to be added to any solution). That is, the spatial and temporal part of the solution are completely independent. Clearly, none of the solutions we have examined so far fit the bill. But we can generally solve for g and f in the following way. Because $T(x, y)$ is a solution to the diffusion equation,

$$\rho c g \frac{\partial f}{\partial t} = k f \frac{\partial^2 g}{\partial x^2}$$

or, separating the functions completely,

$$\frac{\frac{\partial f}{\partial t}}{f} = \frac{k}{\rho c} \frac{\frac{\partial^2 g}{\partial x^2}}{g}.$$

Now, this equation must be true for any value of x and for any value of t. The only way for this condition to hold is if both sides of the equation are equal to the same constant, which we will call ω. We then get two equations:

$$\frac{\partial f}{\partial t} = \omega f \quad \text{and} \tag{4.1}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\rho c}{k} \omega g. \tag{4.2}$$

We now solve these two equations separately. The first is a simple first order linear differential equation with solution

$$f = f_0 \exp(\omega t).$$

The second is a second order differential equation, but we can see by inspection that the simple solution to the equation is

$$g = g_0 \exp(\sqrt{\frac{\omega \rho c}{k}} x),$$

where g_0 is the constant of integration.

The full solution is the product of f and g (plus the constant), so

$$T(x, t) = c_1 \exp(\omega t + \sqrt{\frac{\omega \rho c}{k}} x) + c_2, \tag{4.3}$$

where we have combined the constants f_0 and g_0 into simply $c_1 = f_0\, g_0$.

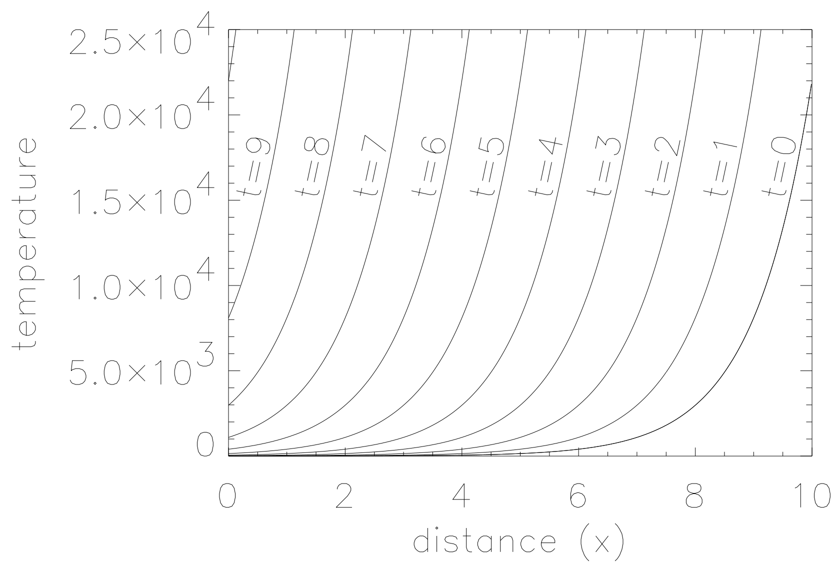

Figure 4.3: A plot of equation 4.3 as a function of time with $\omega = 1$ and $\frac{\rho c}{k} = 1$.

Now that we have found a solution, let's find the problem. We will take the convenient step of setting $c_2 = 0$. What is the steady state solution to this problem? This is a trick question, of course, because there *is no* steady state solution. The positive exponential in t ensures that whatever the solution is, it grow steadily in time. SO what is the solution? Figure 4.1.1 shows the function for a variety of times. Can you tell what (unphysical) situation that this function is the solution for? Think carefully about heat flow in this example, including where it comes from and where it is going.

4.1.2 Imaginary solutions

Equation 4.3 was perhaps not a very useful solution, as nothing in real life increases in temperature exponentially forever. It is possible, however, to *decrease* in temperature exponentially forever, and if we had simply assumed that ω was negative, we would have had nice exponential decay. Let's write this solution out with the numbers explicitly negative:

$$T(x,t) = c_1 \exp(-\omega t + \sqrt{\frac{-\omega \rho c}{k}} x) + c_2. \tag{4.4}$$

In the spatial part of the solution, we now have a square root of a negative number, so we will have an imaginary part, which we can rewrite as

$$T(x,t) = c_1 \exp(-\omega t + i\sqrt{\frac{\omega \rho c}{k}} x) + c_2. \tag{4.5}$$

You have all seen this form of imaginary exponential before and know that

$$\exp(ikx) = \cos(ikx) + i\sin(ikx).$$

Have you ever thought why this is true? There are two ways to convince yourself. The first is to expand $\exp(x)$ in a Taylor expansion. We know then that

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

so

$$\exp(ix) = 1 + ix - \frac{1}{2}x^2 - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 + \dots \tag{4.6}$$

We can now also Taylor expand the sine and cosine function:

$$\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \\
\sin(x) &= x - \frac{1}{6!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots
\end{aligned}$$

Collecting the real and imaginary terms from equation 4.6 above shows us immediately why we are free to say that $\exp(ix) = \cos(ix) + i\sin(ix)$, even though it sounds kind of strange.

All of this is well and good, but we could just as easily used a straight cosine for the solution of the second order differential equation

$$\frac{\partial^2 f}{\partial x^2} = -f$$

because

$$\frac{\partial^2 \cos x}{\partial x^2} = -\cos x.$$

So why do we go through the trouble of these strange imaginary solutions? The main reason is for the behavior of the derivatives. Because $\exp(ix)$ is an exponential, differentiating is trival:

$$\begin{aligned}
\frac{d}{dx}\exp(ix) &= i\exp(ix) \\
\frac{d^2}{dx^2}\exp(ix) &= -\exp(ix) \\
\frac{d^3}{dx^3}\exp(ix) &= -i\exp(ix)
\end{aligned}$$

Differentiating $\cos(x)$ requires more thought

$$\begin{aligned}
\frac{d\cos x}{dx} &= -\sin(x) \\
\frac{d^2\cos x}{dx^2} &= -\cos(x) \\
\frac{d^3\cos x}{dx^3} &= \sin(x)
\end{aligned}$$

and gives us more chances to make mistakes. Using the imaginary solutions is simplest, and, in the end, when we take our solution and try to relate it to reality somehow, we simply need to remember to take the real part of the function (no imaginary temperatures allowed!).

Now let's get back to figuring out what equation 4.5 is all about. The steady state solution of this equation is clearly $T = c_2$, so whatever initial condition you start with decays away to this. The i in the spatial part of the equation tells us that the spatial part oscillates, so we a spatially oscillating temperature which decays away in time. An example would be an infinite block of granite with alternating hot and cold stripes which then decayed to a uniform temperature in time.

This solution gives us the opportunity to discuss interesting behaviors of diffusion. First, note that if we hold ρ and k constant, the decay time is related to the spacing of the sine waves. For example, to describe tightly spaced sine waves, ω needs to be large, which means that the decay will happen quickly. Loosely spaced waves will have a small ω, and the decay time will also be small. Note that this makes intuitive sense! It also makes sense if we think of the diffusion equation as a general equation of time-rate-of-change of temperature. The source of temperature changes is the second-derivative of temperature. So where this second derivative is high, i.e. many curves, the temperature changes quickly. Where the second-derivative is low – few curves – the temperature changes more slowly.

It is also interesting to think of this solution in terms of Fourier analysis. As you all know, any function can be broken up as a series of cosine and sine waves of varying strengths. And, as we noted before, a sum of solutions to the diffusion equation is also a solution to the diffusion equation. So since we know we can solve the diffusion equation for a granite block with sinusoidally varying temperature stripes, we can also solve the diffusion equation for temperature stripes of any arbitrary function, as long as we break the function up into a sum of sines and cosines, for which we know the solution.

When we move to look at the Fourier components, though, an interesting thing happens. What if, for example, we wanted to know the solution when the block of granite has sqaure-wave varying stripes, rather than sinusoidally-varying ones. When we broke the square wave into its Fourier components, we would find that the largest component was a sine wave with the same frequency as the square wave. To make the square wave square, additional higher frequency components would have to be added into the series. But when we go and solve to the diffusion equation, we would find, as above, that these high frequency components would decay away very quickly and in almost no time we would be left with only the lowest frequecy wave left, just as if we had initially started with a sine wave instead of a square wave. What this means is that if I ask you how long it takes for a square wave temperature to decay, you need only figure out how long the lowest frequency term would take to decay, as all of the others would be long gone.

4.1.3 Thermal waves

Now that we have gotten comfortable with imaginary solutions, let's be truly adventurous and try to solve the separated diffusion equation 4.2 where, instead of being positive or negative, we take ω to be imaginary! We will again write the solution our explicityly, taking the imaginary part out of ω and keeping ω strictly real:

$$T(x,t) = c_1 \exp(i\omega t + \sqrt{\frac{i\omega\rho c}{k}}x) + c_2. \tag{4.7}$$

We know now that the $\exp(i\omega t)$ means that there will be a temporally oscillating solution, but what do we do with the $\sqrt{i}$ in the spatiall part of the solution? We simply assume that $\sqrt{i}$ will be some complex number, $a + bi$ and write

$$\begin{aligned}
\sqrt{i} &= a + bi \\
i &= a^2 - b^2 - 2abi.
\end{aligned}$$

Separating the real and imaginary parts, we get

$$\begin{aligned}
a^2 - b^2 &= 0 \quad \text{and} \\
2ab &= 1, \quad \text{so} \\
a &= 1/\sqrt{2} \quad \text{and} \\
b &= \pm 1/\sqrt{2} \quad \text{and}
\end{aligned}$$

Now, substituting back into equation 4.7, we get

$$\begin{aligned}
T(x,t) = \quad & c_1 \exp(i\omega t + (i+1)\sqrt{\tfrac{\omega\rho c}{2k}}x) + \\
& c_2 \exp(i\omega t + (i-1)\sqrt{\tfrac{\omega\rho c}{2k}}x) + c_3.
\end{aligned}$$

Because we are interested in physically interesting solutions, we will set $c_1 = 0$, which means that we don't have any exponential increases, only decays, with depth. Now separating out the real and imaginary parts of the exponent, explicitly taking the real part of the solution, and using $\beta = \sqrt{\frac{2k}{\omega\rho c}}$ we get

$$T(x,t) = c_2 exp(-x/\beta)\cos(\omega t + x/\beta) + c_3 \tag{4.8}$$

This equation is the *thermal wave equation*, and it describes the temperature with depth of a surface subjected to a periodic temperature change. At the surface ($x = 0$), we have the boundry condition that

$$T(0,t) = c_2 \cos(\omega t) + c_3$$

which could describe something like the heating and cooling of day and night or winter and summer. The magnitude decays with depth with a characteristic e-folding length scale (called the *thermal skin depth* of β). Note also, though, that the cosine term has a part that depends on x. This term shows that there is a *phase lag* with depth. At depth, the peak in temperature occurs some time after the peak at the surface. As depth increase, this phase lag also increases.

Note that, as in the previous example we studied, high frequency waves (ω is large) have a small thermal skin depth, so decay very quickly, while low frequency (ω small) waves have a larger skin depth and therefore penetrate much deeper into the surface. This characteristic again shows us that for the basic behaviour, we can simply take the lowest frequency Fourier component and deal with its behavior. Thus, for the surface of the earth, the seasonal wave penetrates $(365)^{1/2} = 19$ times deeper than the diurnal wave. There are those who believe they can measure the temperature and frequency of past ice ages by finding the thermal wave from these events! You'll get to play more with diffusion and thermal waves in the problem set.