Lecture 3

Heat flow

3.1   Conservation of energy

One of the recuring themes in this class and in all of physics is that of conservation of energy. You have all heard this before: energy can be neither created nor destroyed, it can merely change place or form (philosophical quandry: ok, then, where did it come from to begin with?). In problems of gravitation and dynamics, we will always be considering changes from kinetic to potential energy; in chemistry, changes from internal chemical energy to heat. Today, we won't even consider energy changing form, merely flowing from one place to another. The energy that we will deal with is in the form of heat.

Heat energy is completely equivalent to any other type of energy. To prove this, I should be able to give you an example of a system changing, for instance, kinetic energy into heat and then back into kinetic energy with no loss of the initial kinetic energy. In practice, however, I can't give you a good intuitive example of this because, in fact, conversion between heat energy and other types of energy is remarkably inefficient. If everything worked perfectly, however, you should be able to construct a Rube Goldberg-esqe device where, for example, a very insulated bucket of hot water is placed in a cold room under a turbine, which begins to turn because the water heats the air immediately above it, and the hot air rises. The turbine is then attached to a rod which sits inside an insulated bucket of cold water. As the turbine turns, the friction between the water and the rod will heat the water. In the absence of the many real-life losses, the second bucket of water would heat

up to the initial value of the first bucket of water (assuming equal amounts of water in each). Thus heat energy was converted to kinetic energy and then back to heat energy. While this example seems absurd, it is remarkably close to the process by which we get electricity from coal-burning or nuclear sources!

Of course, anyone paying close attention should object that even in a perfect world my above example wouldn't work right because I haven't taken into account the increase in entropy. Objection duely noted.

3.2 The diffusion equation

In the case of heat energy, conservation of energy does not just mean taking all of the heat energy from one object and placing it in another. More important is the case of heat energy flowing within a single object. For example, if one end of an iron rod is placed in a fire, energy will be transferred from the fire to the iron, but then energy will flow through the iron rod to the end not in the fireplace (and if you are holding the iron rod, energy will begin to flow into your hand until you drop the rod). This process of moving heat energy through material is called *conduction*. (Other methods of transferring heat are *convection* and *radiation*). Conduction takes place via a process called *diffusion* which we will now examine.

3.2.1 Pots of water

Consider three equal sized pots of water. The first is held at 50 C, and the last is held at 0 C. The middle pot is originally at 0 C, also. The three pots are touching, so they can transfer heat energy amongst themselves. How does energy flow? Clearly, because the middle and last pots are the same temperature, they will exchange no energy, but because the first pot is hotter than the second, energy will flow from it to heat the second. How much energy? It is an experimentally determined fact, and also an intuitively obvious one, that the heat flow is proportional to the difference in temperature. That is,

$$\frac{dE_12}{dt} = -\alpha(T_1 - T_2)$$
$$\frac{dE_23}{dt} = +\alpha(T_1 - T_2).$$

What is the proportionality constant α in this context? It must have the units of energy/time/temperature and be related to how fast the energy flows.

As soon as the second pot begins to heat, however, it will rise in temperature. The amount of temperature rise is

$$\Delta T = \frac{\Delta E}{mc}$$

where m is the total mass of water in the pot and c is the *specific heat* of the water, in units of energy/mass/temperature, or joule/kilogram/degree Kelvin. That is, c is the number of Joules of energy required to raise one kilogram of a substance by 1 degree K. For water, $c = 4.184 \times 10^3$ J/kg/K. You might be more used to doing heat energies in caolories instead of joules. One calorie is simply 4.184 joules, so conversion between the units is simple. We'll use joules because they are the appropriate MKS unit, which makes converting between heat energy and any other type of energy simpler.

From the equations above, we clearly see that we are going to obtain a differential equation for the temperature of pot 2 as a function of time. Writing the two relevent equations:

$$\frac{dE}{dt} = \alpha(T_1 - T) + \alpha(T_3 - T) \tag{3.1}$$

$$\frac{dT}{dE} = \frac{1}{mc}, \tag{3.2}$$

where now the Es and Ts are a function of time (except that we specified that T_1 and T_3 were held constant) and we have dropped the subscript "2.". If we now multiply both sides of equation 3.1 by eqaution 3.2 we get (by canceling dEs)

$$\frac{dT}{dt} = \frac{\alpha}{mc}(T_1 + T_3 - 2T).$$

This equation is our friend the 1st order linear DE, and we can solve it by inspection to obtain

$$T = \frac{T_1 + T_3}{2} + C \exp(-2\alpha t/mc).$$

Once we constain C so the $T(0) = T_1$, we find that the form of the solution is such that the temperature exponentially approaches the average of T_1 and T_2 with an e-folding time of $mc/2\alpha$. Notice that we could have guessed that the final temperature was going to approach the average of T_1 and T_2: only at this temperature does no net energy flow from pot 2 ($dE/dt = 0$ because as much flows from pot 1 as flows to pot 3).

3.2.2   An iron bar

We now generalize the problem to one where, instead of 3 discreet pots of water, we have a continuous iron bar with a temperature held constant at each end and equal to $T(x, t)$ in between. Note something very important here: this is the first time we have had a function that is a function of two separate variables. Before we have always simply had $N(t)$ or $P(z)$ or $T(t)$. This additional variable is going to add additional complications, as we shall see.

First, let's again calculate the energy flow for an iron bar with temperature $T(x, t)$, mass per unit length of μ, and specific heat of c. For a particular point x, we define the net *flux* of energy to be $f(x, t)$, where by flux, we mean total energy flowing through the point, not the net energy change at a point. Think of the flux as the amount of water flowing in a river, while the net energy change at a particular point is equivalent to the net change in depth of a point in the river as a function of time. In fact, we can write the net energy change at point x as the difference in flux between point x and point $x + \Delta x$:

$$\frac{d\epsilon}{dt} = f(x - \Delta x) - f(x + \Delta x),$$

where ϵ is the energy per unit length of the rod. Again, using the river analogy, if the flow at point x is different than the flow at point $x + \Delta x$, it must mean that the depth of the water at point x is changing, so

$$\frac{d\epsilon}{dt} = -\frac{df}{dx}\Delta x. \tag{3.3}$$

Now we again know the change in temperature associated with this change in energy:

$$\Delta T = \frac{\Delta \epsilon}{\Delta x \mu c}$$

or we can rewrite

$$\frac{dT}{dt} = \frac{1}{\mu c \Delta x}\frac{d\epsilon}{dt},$$

and, substituting into equation 3.3, we obtain

$$\mu c \frac{dT}{dt} = -\frac{df}{dx}.$$

Now we go back to the observed fact that we started off with: that the heat flow between any two points is proportional to the temperature difference between the two points, so

$$f = -k\frac{dT}{dx},$$

where k is now the thermal conductivity, in units of J/sec/K (in general, thermal conductivity will have units of J/sec/K/m, but in our special 1-D case, the length disappears). We now have

$$\mu c\frac{dT}{dt} = \frac{d}{dx}(k\frac{dT}{dx}),$$

which is the one-dimensional *diffusion equation.*

If k is not a function of x (which it might be!), we can rewrite the equation as

$$\mu c\frac{dT}{dt} = k\frac{d^2T}{dx^2}.$$

3.2.3 Partial derivatives

In doing the diffusion equation we have been mathematically sloppy in one subtle way. Each time we had a derivative, say dT/dx, we wrote a full derivative instead of a partial derivative, $\partial T/\partial x$. What's the differnence?

You remember from calculus that, for $f(x, y)$, a partial derivative implies differentiating f by x only where x appears explicitly in f. For example, if $f(x, y) = x^3 + y^2$, then $\partial f/\partial x = 3x^2$. The full derivative, on the other hand, is $df/dx = 3x^2 + 2y(dy/dx)$. Now, as long as x and y are completely independent variables, $dy/dx = 0$ and $df/dx = \partial f/\partial x$, but for any function that is a function of two variables, we have to allow, in principle at least, for the possibility that the variables may be related.

In practice, the situation where the variables are *not* independent will rarely come up. For example, in the diffusion equation above, how could t and x be related in any way?

The single most common time when the difference between full and partial derivatives is crucial, however, is in the case of transforming some calculation to a moving coordinate system (or doing a calculation for a moving object). This situation occurs frequently in fluid mechanics, where the difference between the full and partial derivatives is the physical difference between calculating rates of change of variables that move with the flow or that are

fixed in space. We can even transform our diffusion equation into one where the full/partial difference matters by considering, instead of a fixed bar, a moving bar. Let's assume the same bar as before is now moving with velocity v, so that $x = x_0 + vt$. Now x and t are explicitly linked! The fact that bar is moving does not change the solution to the equations, as long as we solve them in the moving frame, but if we solve them in a stationary frame, the formal solution will end up being not $T(x,t)$, but $T(x_0 + vt, t)$, which will have a different functional form.

In general, when you have a function of two or more variables, *all* derivatives will be partial derivatives unless there is something peculiar going on. So, when in doubt, make it a partial.

3.2.4   3D

We can easily generalize the 1D diffusion equation to three dimensions (again, assuming the k is constant):

$$\rho c \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = k\nabla^2 T,$$

where now k has the standard units of J/sec/m/K and we have the standard diffusion equation.

If you've seen the notation before, then you recall that

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

If you haven't seen the notation before then you have now: ∇ is called "del" and ∇^2 is called "del squared." It is nothing more than a shorthand notation.

Sometimes in such an equation, we do not care about the time-variable behavior of the solution, only the steady-state solution. Steady state means, of course, that all time derivatives must be zero, so the diffusion equation simply becomes

$$\nabla^2 T = 0$$

(this equation comes up frequently in physics and is called *Laplace's equation*), or in one-dimension

$$\frac{\partial^2 T}{\partial x^2} = 0$$

The one-dimensional equation is easily solvable by simply integrating twice:

$$\begin{aligned}
\int \frac{\partial^2 T}{\partial x^2} &= \int 0 \, dx \\
\frac{\partial T}{\partial x} &= C_1 \\
T &= C_1 x + C_2
\end{aligned}$$

Note something different going on here: we suddenly have *two* constants of integration rather than just one. This situation is a consequence of the fact that we have a second-order differential equation. For each order of the differential equation, a new constant of integration appears (simply because we need that many integrations to solve the equation). How do we handle the two constants of integration? Again, with two boundry conditions. In the case of our example, we stated that the ends of the iron rod were held fixed at temperatures of 100 and 50 C. If the iron rod is 1 meter long, then the steady-state solution is

$$T(x) = -50x + 100,$$

where x is in meters and T is in degrees C. The very reasonable seeming solution is that the temperature changes linearly between the two fixed end points.

(As an aside, why did we never consider the steady state solution to the population equation? Clearly, if we have an equation of the form $dN/dt = -\alpha N$ and we seek a solution with $dN/dt = 0$, our only solution is $N = 0$. Indeed this is a steady-state solution (if you start with zero population you stay with zero population), but certainly not a very exciting one!)

The full three-dimensional case is harder to solve simply because we have many more possible constaints to satisfy.

3.3   Solutions to the diffusion equation

Though the diffusion equation is not particularly complicated looking, its full solution can require some fairly sophisticated techniques. In fact, unlike the case of the linear first order DE, no general solution to the problem exists at all. To solve the equation will in general actually mean just finding one of the known solutions that actually fits the boundry values of the problem.
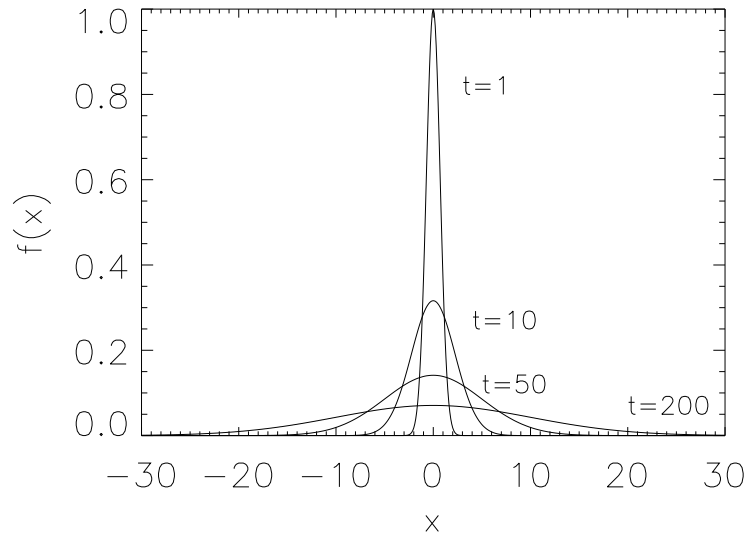
Figure 3.1: A solution to the wave equation. This solution can be thought of as the temperature response to an instantaneous energy pulse, say a laser blast at one spot.

Let's then examine some of the known solutions of the equation. And, of course, at any time we could simply use numeric methods to solve the problem.

3.3.1   A heat pulse

Consider the function

$$T(x,t) = (4\pi kt)^{-1/2} \exp(-x^2/4kt). \tag{3.4}$$

Plug this into the diffusion equation and you will find that it is a valid solution (and not one that you should have guessed ahead of time!). Let's examine, just for interest's sake, the general behavior of the solution. In figure 3.3.1 we plot $f(x,t) = t^{-1/2} \exp(-x^2/t)$ for various values of t. The solution is one where an initially highly confined temperature pulse (say a spot that has been blasted by a laser) diffuses outward through the surrounding medium.

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Because heat is neither gained nor lost anywhere in this problem, the total heat must be conserved. The total heat is

$$E = Tc = c \int_{-\infty}^{+\infty} T(x,t) \, dx.$$

You should recognize equation 3.4 as one of a standard *gaussian distribution*, and recall that its integral over $\pm\infty$ is a constant, and, in fact, we have defined the constant in front of equation 3.4 so that the integral equals 1. Thus for any time t, the integral of equation 3.4 equals 1. As the pulse spreads out in time, the total change in temperature caused by the pulse stays constant (actually the total heat stays constant, and assuming c is uniform, the total temperautre deviation remains constant).

This solution to the diffusion equation can be used any time the initial conditions warrant the solution. When does this happen? Only when the temperature distribution at some time t is actually equal to equation 3.4. So what good is this to anyone? The short answer is "not much." The main point is just to illustrate an interesting feature of the solution, that of the constant diffusing heat. The long answer, however, is that we can use this solution to find another solution which *is* important for general use.

3.3.2 Summing solutions

If f(x,t) is a solution of the diffusion equation and g(x,t) is an independent solution then

$$\frac{\partial f}{\partial t} = k \frac{\partial^2 f}{\partial x^2} \quad \text{and}$$
$$\frac{\partial g}{\partial t} = k \frac{\partial^2 g}{\partial x^2}.$$

We can add these two equation together to get

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} = k \frac{\partial^2 f}{\partial x^2} + k \frac{\partial^2 g}{\partial x^2}$$

or simply

$$\frac{\partial (f+g)}{\partial t} = k \frac{\partial^2 (f+g)}{\partial x^2}.$$

Now we see that $g(x,t) + f(x,t)$ is also a solution to the diffusion equation.

This example illustrates a general solution, that a sum of any number of solutions to the diffusion equation is also a solution to the diffusion equation. But if a sum of solutions is a solution, it should also be true that an integral of solution, which is just a sum of infinitely finely spaced solutions, should also be a solution.

Let's then integrate equation 3.4 to get a new solution to the diffusion equation:

$$T(x,t) = (4\pi k t)^{-1/2} \int_0^x \exp(-x'^2/4kt)\, dx'$$

which reduces to

$$T(x,t) = \frac{1}{\sqrt{\pi}} \int_0^u ds \exp(-s^2) = \frac{1}{2}\mathrm{erf}\, u$$

or

$$T(x,t) = T_0 \mathrm{erf}(x/(4kt)^{1/2})$$

where we have introduced the *error function*, $\mathrm{erf}(x)$. The error function is a standard function, just like sine, cosine, or log, that is tabulated in books, included on calculators, and incorporated in computer languages (in MATLAB, type: `e=erfc(1.)`, for example, and you will get `0.842701`). Figure 3.3.2 shows what the error function looks like. The two important properties, which make the error function a good function to try to use to match initial conditions, can be seen in the figure,

$$\mathrm{erf}(0) = 0 \quad \lim_{x \to \infty} \mathrm{erf}(x) = 1.$$

When is the error function a useful solution to the diffusion equation? Anytime the initial condition along some boundry of the diffusion equation is equal to a constant. We'll solve such a problem in the next lecture.
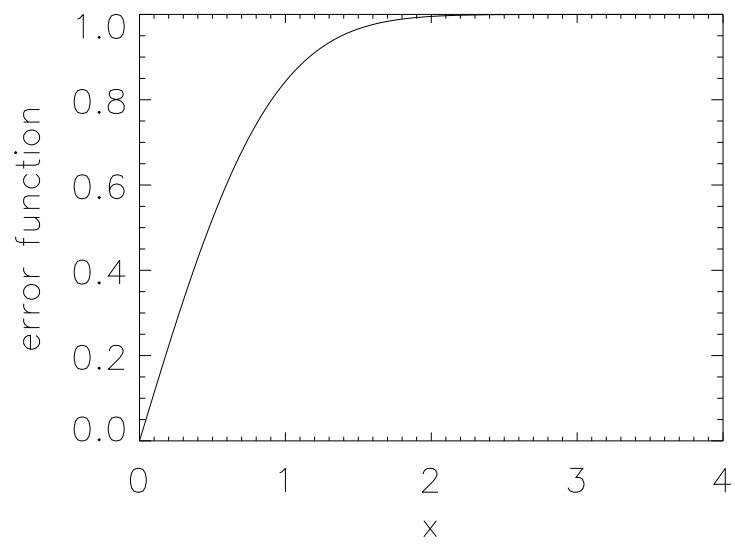
Figure 3.2: The error function.