

Lecture 2

Applications of 1st Order DEs

2.1 Hydrostatic equilibrium

One of the most important equations for understanding the structure of the earth, the planets, the atmosphere, the ocean, and stars is the equation of *hydrostatic equilibrium*. The word “hydrostatic” in this context means “stationary fluid;” with this equation we will be able to solve for the structure of any collection of a stationary fluid-acting substance, be it the air in the atmosphere, the water in the ocean, or the material in the earth’s mantle. Hydrostatics does not apply in situations where the materials are either non-liquid or non-stationary, like, for example, a rock or a mountain, which is definitely non-fluid, or a river, which is non-stationary. In fact, most real objects are non-stationary. Atmosphere, mantles, and oceans all have convective motions, meaning that the assumption of hydrostatics does not hold. But, as we will, see, hydrostatic equilibrium is a good first approximation to all of these things.

To solve for the structure of the atmosphere (or ocean or mantle), we would like to know the pressure at every height z . Atmospheric pressure, with units of force per area, is simply equal to the weight of all of the air above a unit area. On the earth, the average surface atmospheric pressure is 1 bar or 1.013×10^5 Pascals (force/area: $\text{kg}/\text{m}/\text{sec}^2/\text{m}^2$) or about 14.5 lbs/in². (Question to think about: if the air pressure is 14.5 lbs/in², how come a 8 by 11 inch piece of paper does not weigh 1200 lbs?) What is the pressure as we go up in altitude?

Let’s start at height z (possibly the surface) with pressure $p(z)$. If we

move to a slight greater height, $z + \Delta z$. the weight of atmosphere is decreased by the weight of the atmosphere between z and Δz , so

$$p(z + \Delta z) = p(z) - g\rho(z)\Delta z,$$

where g is the earth's gravitational pull, $\rho(z)$ is the average density of material between z and $z + \Delta z$, and $g\rho(z)\Delta z$ is the weight of material between z and $z + \Delta z$ (note the units: g is an acceleration, m/sec², ρ is a density, kg/m³, and Δz is a length, m, so the overall expression has units of m/sec² kg/m³ m, or kg/m/sec³/m², a pressure, as expected).

Now, rewrite the equation as

$$\frac{p(z + \Delta z) - p(z)}{\Delta z} = -g\rho(z),$$

and if we take the limit as $\Delta z \rightarrow 0$, the left-hand side of the equation becomes the definition of a derivative, so we have

$$\frac{dp}{dz} = -g\rho(z), \tag{2.1}$$

which is the *equation of hydrostatic equilibrium*.

This equation is a first-order DE, but we can't yet say whether it is linear until we know the function $\rho(z)$. In the simplest case, that of an incompressible fluid, the density is constant, and we get a particularly simple form of the equation:

$$\frac{dp}{dz} = -g\rho$$

which is immediately separable and integratable and gives

$$p = -g\rho z + p_0.$$

That is, the pressure decreases linearly with height from its initial value p_0 . Water is *almost* this simple, but it is slightly compressible, and its density *is* a function of temperature. But, as a first approximation, this equation holds nicely. As an example, we know that water had a density of something near 1 g/cm³, or 1000 kg/m³, so if the pressure at the top of a swimming pool is simply the 10⁵ Pascal atmospheric pressure, the pressure at the bottom of a 10 meter swimming pool (admittedly a deep pool) is

$$p = 9.8\text{m/sec}^2 \times 10^3\text{kg/m}^3 \times 10\text{m} + p_0 = 9.8 \times 10^4 \text{ Pascals} + p_0 \approx 2p_0.$$

Thus, about 30 feet of water weights the same as all of the atmosphere above your head.

In general, though, to solve the equation of hydrostatic equilibrium, we need an *equation of state* which relates p , ρ , and T , the temperature. In general, the equation of state will depend on the specific materials present. Much of the research into the properties of the earth's interior could be simplistically thought of as trying to determine the equation of state for these materials in these extreme pressure conditions.

We already know one simple equation of state, though. This equation is the *ideal gas law*. You may have learned it in chemistry in the context of the relationship between pressure, volume, and temperature in, say a ballon. In this case it was written as

$$PV = nRT$$

where V is the volume of the ballon, n is the number of moles of gas in the ballon, and R is the universal gas constant. For free gas, not in an enclosure, we rewrite n/V as ρ/μ , where μ is the mean molecular weight of the gas, to get

$$p = \frac{\rho}{\mu}RT.$$

The ideal gas law works well for gases at moderate pressures and densities (those not greatly larger than atmospheric, say). At higher pressures, as in the interior of Jupiter or the sun, and at higher densities, such as in liquids, significantly more complicated equations must be used.

Putting the ideal gas equation into the equation of hydrostatic equilibrium (equation 2.1), and remembering that we have still not specified how the temperature changes with height, we get

$$\frac{dp}{dz} = -\frac{g\mu}{RT(z)}p.$$

We know how to solve these types of linear 1st order DEs now. We could go for the full general solution, but this equation is simpler than most: the variables are immediately separable into ps and zs :

$$\frac{dp}{p} = -\frac{g\mu}{RT(z)}dz$$

and integratable like the population equation

$$\ln p = \int -\frac{g\mu}{RT(z)} dz + C$$

or, as before

$$p = p_0 \exp\left(-\int \frac{g\mu}{RT(z)} dz\right)$$

Just like in the population equation, were we had an e-folding time scale, the factor in the exponent in this problem is the reciprocal of an e-folding height, called the *scale height* of the atmosphere:

$$H = \frac{RT(z)}{\mu g}.$$

For an atmosphere where the temperature changes with z (which is all of them), the scale height is somewhat ill-defined; it is the height at which the pressure would be decreased by $1/e$ *if the temperature were held constant*. Though not a perfect measure, the scale height is still an extremely useful number to tell you about general length scales in an atmosphere.

An important special case is that of an *isothermal atmosphere* for which T is constant, in this case, the integral is solvable:

$$\int \frac{\mu g}{RT} dz = \int \frac{dz}{H} = \frac{z}{H} + C,$$

so the pressure varies with height simply as

$$p(z) = C \exp(-z/H)$$

where we again get C from the initial conditions. If we know, for example that the pressure at $z = 0$ is p_0 , the solution becomes

$$p(z) = p_0 \exp(-z/H).$$

In most planetary atmospheres T varies slowly with height and thus, so does the scale height, so this equation almost holds. Essentially all atmospheres have an isothermal region where T is approximately constant with height, so therefore H is also constant in this region. On the earth, this region occurs between about 10 and 15 km from the surface.

In general, there is no way to specify the temperature of the atmosphere except through either direct measurement or detailed calculation of the effects of the incoming sunlight (direct heating and photodissociation) and the outgoing thermal emission (heating). We have empirically found that for the lowest part of the Earth's atmosphere (the "troposphere"), the temperature

decreases 6.5 K for every kilometer of gained elevation. This fact turns out to be a straightforward result from thermodynamics, that, in the absence of other effects, air cools as it expands and heats as it compresses. (When you pump up your bicycle tire, the pump head gets very hot from compressing and therefore heating the air. Likewise, if you let air out of the tire, it suddenly expands and cools.) So we can write

$$T(z) = T_0 - kz$$

and the integral in the equation of hydrostatic equilibrium becomes

$$\int \frac{\mu g}{R(T_0 - kz)} dz = \frac{\mu g}{R} \int \frac{-dy}{ky}, \text{ where } y = T_0 - kz$$

Full solution of this equation will be part of your problem set.

2.2 More 1st order DEs

First order DEs occur anytime we have a straightforward time-rate-of-change problem. In this case, we can usually make up the DE on the spot. As we discussed in the last lecture, such equations can generically be written as

$$\frac{dx}{dt} = \text{source of } x - \text{sink of } x.$$

From this generic equation, we can often simply make up our own differential equations for many physical processes in the world.

2.2.1 People

As a quick example, consider a population of people who breed and die in an environment with infinite resources (no over-population starvation problems, for example). The source of new people is births, which should be proportional to the number of people in existence, while the sink of people should be deaths, which is also proportional to the number of people. We can write the equation for the number of people, then, as

$$\frac{dP}{dt} = \beta P - \delta P,$$

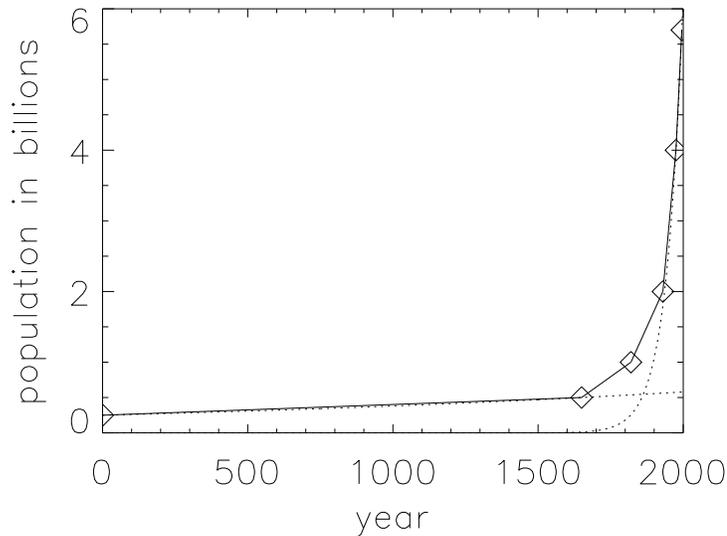


Figure 2.1: The human population of the earth for the past 2000 years. The dashed lines show attempts to fit exponential curves to the first two and the last two points. Note that the growth is super-exponential!

where P is the population, β is the reciprocal of the birth-rate e-folding time, and δ is the reciprocal of the death rate e-folding time. This equation is the simple population equation that we have already solved. The answer is

$$P = P_0 \exp[(\beta - \delta)t].$$

If $\delta > \beta$ (the death rate is larger than the birth rate) the exponent will be negative and the population will exponentially decline. If $\beta > \delta$ (birth rate is larger than death rate) the exponent will be positive and the population will experience exponential growth. Figure 2.2.1 shows the population of humans since the year 1 A.D. Which is larger, β or δ ?

Of course, we already knew without being shown that human population is increasing because the birth rate is larger than the death rate. How well does the population increase fit the population equation? To find out, we try to fit an exponential growth curve to the population. It takes two points to define an exponential growth curve (because we have two unknowns:

P_0 and the growth time-scale τ). We first, then, solve for the exponential growth from AD 1 (population 0.25 billion) until AD 1650 (population 0.5 billion) and determine that the e-folding is 2380 years. If we use the last two measured points, however ($P(1975) = 4$ billion; $P(1995) = 5.7$ billion) we get an e-folding time of only 56 years! Clearly our simple population equation isn't holding; the human population is increasing super-exponentially. How can this happen? Again, we already know the answer. In our population equation, we assumed that $\beta - \delta$ was constant, that is the birth and death rates didn't change. But death rates have decreased dramatically while birth rates have not changed nearly so much. The result is that the e-folding time for population increase continues to decrease and the population shoots up faster and faster.

2.2.2 Rabbits & Foxes

As a further example, let's now consider rabbits, instead of humans. Rabbits are famous for reproducing exponentially, and their population in general follows the population equation

$$\frac{dR}{dt} = \beta_r R - \delta_r R,$$

where now β_r and δ_r are the rabbit e-folding birth and death reciprocal time scales. In the absence of predation or starvation, the rabbits will reproduce like rabbits. Unfortunately for the rabbits of this example, however, there are also foxes, who like to eat the rabbits, giving another sink for rabbits. What exactly is the mathematical form of this sink? It is clearly proportional to the number of foxes, so we might try a sink of $-\epsilon_f F$ (where ϵ_f is the reciprocal time-scale for foxes to eat $1/e$ of the rabbits, and F is the population of foxes), which would then give

$$\frac{dR}{dt} = \beta_r R - \delta_r R - \epsilon_f F.$$

A little thought shows that this equation cannot be correct because, for example, when there are no rabbits left, the birth and deaths go to zero also (as they should), but our fox-eating-factor stays high, since it is only proportional to the number of foxes. The number of rabbits then becomes *negative* which, of course, makes no sense. Clearly, we need to also multiply

the fox-eating-factor by the number of rabbits to get

$$\frac{dR}{dt} = \beta_r R - \delta_r R - \epsilon_f F R. \quad (2.2)$$

Having a factor proportional to the product of two populations is a general form that comes about anytime you are dealing with interactions between two things, be it collisions of particles, rates of chemical reactions, or number of rabbits eaten.

We can now also write down an equivalent equation for the number of foxes. Foxes are born and foxes die and let's assume that the more rabbits there are to be eaten the more foxes will be born (or the fewer foxes will die: these are mathematically equivalent):

$$\frac{dF}{dt} = \beta_f F - \delta_f F + \epsilon_r F R,$$

where now ϵ_r is related to the increase in fox birth rate.

We have here a set of *coupled* differential equations. Clearly the solution to one depends on the solution to the other. Such equations can lead to interesting time-variable behavior as the two different populations feed back to one another. Analytic solutions of these types of equations have not been discussed yet, but they are quite simple to solve numerically (see next section). As an example, in Figure 2.2.2 we've plotted the number of rabbits and number of foxes as a function of time for a particular set of choices of β s, δ s, and ϵ s and initial conditions (initial conditions here will be the initial number of foxes and rabbits). Look at the feedback going on here: the rabbit population initially crashes because there are too many foxes around. Soon thereafter, the fox population crashes because there are no rabbits to eat. But with few foxes left, the rabbit population begins to soar again. Soon the foxes follow, and the rabbits begin to decline. The MATLAB program used to solve these equations can be found on the Ge 108 web page.

2.2.3 Smog

As a final example, let's consider the amount of various species of smog in the atmosphere as a function of time of day. Figure 2.2.3 shows the amount of ozone, and NO_2 in Pasadena on 3 October 1997 as a function of time of day. Notice that ozone varies with time of day while the other two species do not. Can we write a differential equation to reproduce this behavior? For

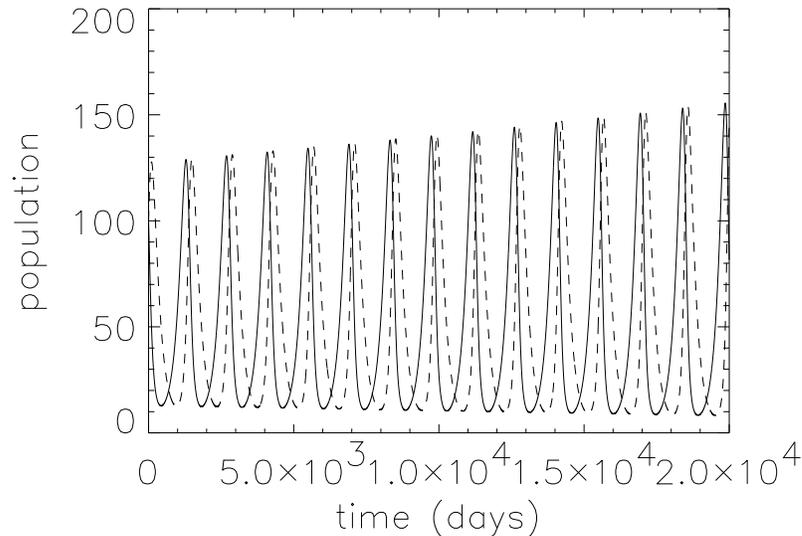


Figure 2.2: The number of rabbits (solid line) and the number of foxes (dashed line) found from the numeric solution of equations 2.2 and 2.2.2. The initial conditions were chosen to be $R_0 = 100$, $F_0 = 100$, $\beta_r = 0.01$, $\delta_r = 0.005$ (note that the rabbit population would grow exponentially in the absence of foxes), $\beta_f = 0.005$, $\delta_f = 0.01$, (the fox population would decline exponentially if there were no rabbits around to eat), and $\epsilon_r = \epsilon_f = 0.0001$. All units are days⁻¹.

all three species, the source is photochemistry. Let's assume that the photochemical rates are directly proportional to the amount of sunlight, which we will approximate as

$$\text{sunlight} = 1 - \cos(t/t_d),$$

where t_d is the time for one cycle, or 1 day/ 2π . Note that we have to add the constant to the beginning of the equation to make sure that we never get negative sunlight! The smog source rate will then just be

$$\text{source} = \alpha(1 - \cos(t/t_d)).$$

We can assume that the sink for smog is chemical decay to less noxious species, so that the decay is just given by an e-folding decay constant, τ . The

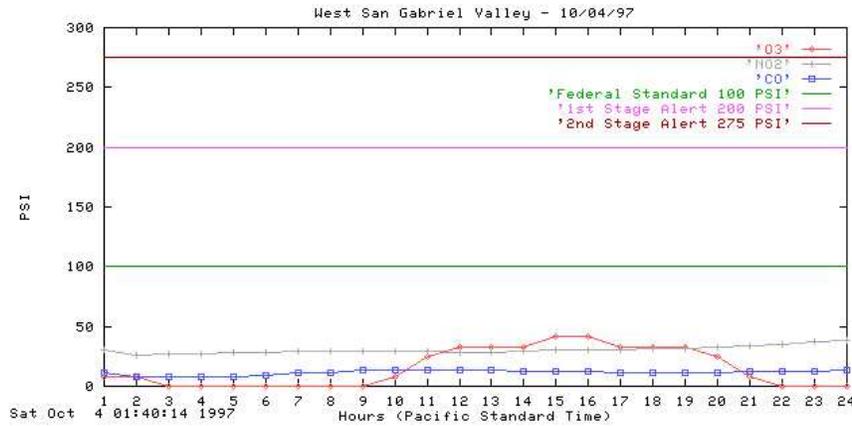


Figure 2.3: Smog in Pasadena on 3 October 1997. All species are photochemically produced, but ozone has a much shorter half-life than the other two species, so it more strongly reflects the sunlight source.

value of τ will be different for each different chemical species in the smog.

The smog equation then becomes:

$$\frac{dS}{dt} = \alpha(1 - \cos(t/t_d)) - S/\tau.$$

This equation is a linear first-order DE and is solvable by the general method of using an integrating factor, as we discussed in lecture 1. If we multiply by $\exp(t/\tau)$, the equation reduces to

$$\exp(t/\tau)S = \int \alpha \exp(t/\tau)(1 - \cos(2\pi t/t_d)) dt + C$$

which, using the identity

$$\int \exp(ax)[\cos(bx)] dx = \frac{\exp(ax)}{(a^2 + b^2)}[a \cos(bx) + b \sin(bx)]$$

becomes

$$S = \alpha\tau - \frac{\alpha}{1/\tau^2 + 1/t_d^2} \left[\frac{1}{\tau} \cos(t/t_d) + \frac{1}{t_d} \sin(t/t_d) \right] + C \exp(-t\tau)$$

Now let's see if this equation can reproduce the general behavior of Figure 2.2.3. If $\tau \gg t_d$ we have a situation where the e-folding decay time of the smog species is long compared to one day. We would then expect to see little daily variation in the solution. In fact, for the limit $\tau \rightarrow \infty$, we get

$$S = \alpha\tau - \alpha t_d \sin(t/t_d),$$

which, since $\tau \rightarrow \infty$ becomes simply

$$S = \alpha\tau,$$

a constant, as expected. For the opposite case ($\tau \ll t_d$), the smog decays quickly after being produced, so it should just track the available sunlight. In the limit the $\tau \rightarrow 0$, we have

$$S = \alpha\tau(1 - \cos(t/t_d)),$$

a small, but oscillating population. Figure 2.2.3 shows the exact solutions for the cases where $\tau = 1$ hour and $\tau = 240$ hours. Our simple equation does a nice job of reproducing the behavior of the real smog concentration in Pasadena.

2.3 Numeric methods

2.3.1 Euler's method

We have learned how to generally solve many linear first-order differential equations, but, in practice, many of the 1st order differential equations you come up against will not, in fact, be linear. Or they may be linear but they have an unintegratable solution. In these cases, it is best to resort to numeric methods on the computer to get your solution.

Entire classes exist in numeric methods of solving integrals and differential equations, and many clever tricks exist which can save you much computer time and result in much higher accuracy. We are going to ignore all of that and just consider a brute force solution to these equations.

Consider, again, the equation

$$p(z + \Delta z) = p(z) - g\rho(z)\Delta z. \tag{2.3}$$

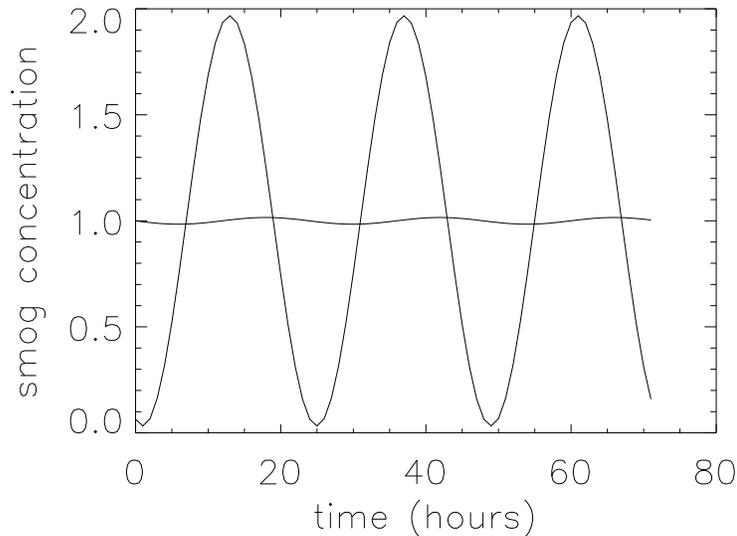


Figure 2.4: Solutions to the smog equation. For a small e-folding destruction time (1 hour, in this case), the species concentration follows the sunlight source. For a long e-folding destruction time (10 days), the species concentration remains essentially constant with time.

Previously, we took the limit of this equation as $\Delta z \rightarrow 0$ and turned it into a differential equation. Now, instead, let's deal with this equation directly.

If we know p at any z and have a function for $\rho(z)$ (an equation of state would do), we can directly solve equation 2.3 for $p(z + \Delta z)$ by assuming a small value for Δt .

For example, if we take the case of an isothermal atmosphere and write equation 2.3 as

$$p(z + \Delta z) = p(z) - \Delta z p(z)/H$$

and assume an initial condition of $p(0) = 10^5 \text{ P}$ and scale height of 10000 m, we could estimate $p(1 \text{ meter})$ as

$$p(1\text{m}) = 10^5 \text{P} - 10^5 \text{P}/10^4 \text{m} \times 1\text{m} = 9.999 \times 10^4 \text{P}.$$

But we now know $p(1 \text{ meter})$, so we could repeat this equation to find $p(2$

meter). We could continue solving for each altitude individually until we have solved completely for pressure as a function of altitude.

This method of solution of a differential equation is called *Euler's method*, and, while it is neither the most accurate or efficient algorithm available, it is the conceptual basis for all other numeric solutions.

2.3.2 Computer implementation

Euler's method is simple to implement using almost any computer language. All that is required are arrays to store the values of p and z and an ability to loop. We'll be using MATLAB examples throughout this class, but these could easily be transformed into whatever your favorite programming language is. Consider the following program (where text following a ";" is a comment):

```
% MATLAB program to solve the isothermal equation of hydrostatic equilibrium
deltaz=1; % use 1 meter distance steps
H=10000; % define scale height to be 10km
p=zeros(30000); % do 30,000 steps of deltaz
z=(0:30000)*deltaz; % z now contains 30000 heights
p(1)=1. % boundary condition
for n:30000 % loop the variable n from 3 to 30000
p(n)=p(n-1)-p(n-1)/H*deltaz; % implement Euler's method!
end

plot(z,p)
xlabel('height (meters)')
ylabel('pressure (bar)')
end
```

This method obviously relies on choosing Δz to be small, but how small? Clearly, in implementing this numeric method, we are assuming that the function $p(z)$ is linear on the length scale of Δz . This assumption holds nicely as long as Δz is smaller than the length scale over which $p(z)$ changes, which we know to be H . So we expect that this method should work well for $\Delta z \ll H$.

In fact, look what happens if we make $\Delta z = H$, which we expect to be too big. We then have

$$p(\Delta z) = p(H) = p(0) - p(0)/H \times H = 0!$$

We know, however, that $p(H)$ should be $1/e$ rather than 0.

If making Δz too big causes problems, why not just set Δz to something ridiculously small, like one millimeter? The reason is simply computational efficiency. The number of calculations (and thus computer time) needed to solve the equation numerically is inversely proportional to Δz , so we would like the largest value of Δz that we give us a solution of the desired accuracy. The problem set will allow you the opportunity to play with different step sizes to see what affect they have on the accuracy of the final solution. More complicated numeric integration schemes have been developed in an attempt to get higher accuracy in a smaller amount of time. We will explore some of these more complicated schemes in the weeks to come.