Higher-mode ambient-noise Rayleigh waves in sedimentary basins

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SUMMARY

We show that higher modes are an important component of high-frequency Rayleigh waves in the cross-correlations over sedimentary basins. The particle motions provide a good test for distinguishing and separating the fundamental from the first higher mode, with the fundamental mode having retrograde and the first higher mode having prograde motion in the 1–10 s period of interest. The basement depth controls the cut-off period of the first higher mode, which coincides with a rapid increase (over period) in the particle-motion ellipticity or $HV$ ratio of the fundamental mode. The strong higher mode we observed is not only due to the low-velocity sedimentary layer but also due to the noise sources with significant radial component such as the basin edge scattering. It is important to correctly identify the mode order when inverting the dispersion curves because misidentifying the higher mode as fundamental will lead to an anomalous high $V_{5V}$ velocity.

Key words: Interferometry; Surface waves and free oscillations; Wave propagation.

1 INTRODUCTION

Determining the basin structure is important for evaluating the potential seismic hazard because basins trap and amplify strong motion energy (Olsen 2000; Komatitsch et al. 2004). The Los Angeles Basin is a typical example of this, where there have been many studies of its structure (Hauksson & Haase 1997; Fuis et al. 2001; Süss & Shaw 2003; Tape et al. 2009; Lee et al. 2014; Shaw et al. 2015), using data from both passive and active seismic experiments, as well as well-logs and reflection data from the oil industry. The culmination of these efforts has been the creation of a series of steadily improving Community Velocity Models (Kohler et al. 2003; Plesch et al. 2011; Shaw et al. 2015), which are used among other things, in the simulation of ground motions from scenario earthquakes.

Methods based on ambient noise cross-correlation (Shapiro et al. 2005; Yao et al. 2006; Lin et al. 2008, 2014) have been an important development in building basin models. With a dense recording array, both high-frequency surface waves (Lin et al. 2013; Shirzad & Shomali 2014; Fang et al. 2015) and body wave signals (Nakata et al. 2015) can be extracted, and used to determine the structure. The ellipticity of the Rayleigh-wave particle motion is also being measured (Savage et al. 2013) which can be used to infer the depth of the basin. The generation of empirical Green’s functions by correlation can be used to generate scenario earthquakes, which when compared to numerical simulations can be used to test the basin models (Denolle et al. 2013, 2014).

One problem that arises when using ambient-noise surface waves to determine structure of basins is the presence of higher modes. The standard analysis generally assumes that the fundamental mode is dominant in the vertical component cross-correlations. While this is generally true for regional scale surveys, there are cases where the higher-mode Rayleigh wave can also be strong (Savage et al. 2013; Rivet et al. 2015). A misidentification of the mode can lead to a higher $V_{5V}$ and incorrect anisotropy estimates ($V_{5V}$ versus $V_{SH}$), as well as incorrect amplitude information.

In this paper, we analyse surface waves generated by ambient-noise correlations using data from a dense broad-band array that was deployed across the Los Angeles Basin. The density of the array allows us to clearly see the modes and to measure their properties. The actual model of the basin that is determined will be the subject of another paper (Ma & Clayton 2016) where the surface wave dispersion curves are combined with receiver functions to measure the shape and velocity structure of the Los Angeles Basin.

Fig. 1 shows the location of the dense array comprised of 73 three-component broad-band seismometers, and the CVM-H model along the profile. Of particular interest in this paper are 44 (of 73) stations that are deployed in a linear array with $\sim 1$ km interstation distance. They were operational from September to November 2014, with an average recording time of about 40 d. This ‘high-density short-duration’ experiment, named the ‘Los Angeles Syncline Seismic Interferometry Experiment’ (LASSIE), turns out to be effective for the ambient-noise studies and may serve as a prototype experiment that will allow basins to be covered by this type of low-cost short-duration survey.

2 MULTI-COMPONENT CROSS-CORRELATIONS

Following methods of Bensen et al. (2007) and Lin et al. (2014), we perform cross-correlations between all three components of each pair of stations. The Green’s function can be approximated by the
negative of the time derivative of the cross-correlation with the positive and negative time lags stacked. Either of the two stations can be the ‘virtual source’, and the other is the ‘receiver’, with the source or receiver component being the one used in the cross-correlation. In the following, we take the cross-correlations between N116 and all the linear-array stations (yellow dots in Fig. 1) as examples. The Rayleigh-wave emerges from the tangential-component (T–T) cross-correlations. Clear Love wave signals are observed along the 1-s period profile in Fig. 2(a) and 5-s period profile in Supporting Information Fig. S1, except in the 1-s period for those stations located at a distance beyond 30 km. In Fig. 2(b), we take the cross-correlation between N116 and A138 as an example. It shows that the noise in the 1-s period cross-correlation is persistent regardless of the number of days used in the cross-correlation. Note that the Puente Hills is located at a distance beyond 30 km. In Fig. 2(b), we take the cross-correlations between N116 and A135 as an example, and the results are shown in Figs 4(a) and (b). N116 is located in the basin while A135 in the Puente Hills (Fig. 1). In Figs 4(a) and (b), we also annotate the pair of cross-correlations used for each particle motion measurement, and the corresponding force direction at the virtual source. For example, using Z–Z and Z–R cross-correlations, which correspond to a Z-directed force at N116, we can measure the particle motion at A135. Using the Z–Z and R–Z cross-correlations, which correspond to a Z-directed force at A135, we can measure the particle motion at N116. Note that the sign of the R–Z cross-correlation needs to be inverted since the R direction (from A135 to N116) is opposite to that in the cross-correlation (from N116 to A135). We use a time window of one wavelength centred at the peak of the envelope to measure the particle motion. The particle motion...
ellipse is colour shaded with the time to show the direction of motion.

Figs 4(a) and (b) show the particle motions measured for different modes at different periods. The measurements with different source directions (Z or R) generally give similar results. There are gaps in our measurements, because we only measure when observe clear signals arriving at same time (same mode) in the cross-correlation pair. We observe that: (1) higher mode shows prograde particle motion, while the fundamental mode is retrograde, and (2) the higher mode disappears around 4 s, and above this ‘cut-off period’ the fundamental mode in the basin (N116) shows high ellipticity or H/V ratio.

We also note that the particle motion ellipse is not strictly upright as expected for a Rayleigh wave. This phenomenon is not specific to this study and is also shown in Lin et al. (2014, fig. 2 therein). In Section 4, we will show that the tilted ellipse is readily explained by body wave interference.

3 RAYLEIGH WAVES IN A BASIN MODEL

Some of the Rayleigh-wave observations can be explained with a 1-D basin model. The Rayleigh-wave Green’s function in a 1-D layered structure is (Aki & Richards 2002):

$$ G = \begin{bmatrix} G_{RR} & G_{RZ} \\ G_{ZR} & G_{ZZ} \end{bmatrix} $$

$$ = \sum_n \frac{1}{8cU I_1} \begin{bmatrix} r_1(z)r_1(h) & -ir_1(z)r_2(h) \\ ir_2(z)r_1(h) & r_2(z)r_2(h) \end{bmatrix} \times \left( \frac{2}{\pi k_n \Delta} \right)^{1/2} \exp \left[ i \left( k_n \Delta + \frac{\pi}{4} \right) \right]. \tag{1} $$

where $n$ denotes the $n$th mode, $h$ is the depth of the point source, $z$ is the depth of the receiver, $\Delta$ is the distance between source and receiver; $r_1$ and $r_2$ are the horizontal and vertical displacement eigenfunctions (of the $n$th mode), $k_n$ is the wavenumber of the $n$th mode, $c$ and $U$ are the phase and group velocity (of the $n$th mode) respectively, and $I_1 = \frac{1}{8} \int_0^\infty \rho (r^2_1 + r^2_2)dz$ is the energy integral. What we observed are the fundamental ($n = 0$) and the first higher mode ($n = 1$). Note that the higher mode discussed in this paper refers to the first higher mode.

We see that for each mode, the particle motion ellipse is determined by $r_1(z = 0)$ and $r_2(z = 0)$ at the receiver, and the dependence of the wave amplitude on the source depth or direction (R or Z) is also controlled by the eigenfunctions $r_1(h)$ or $r_2(h)$. In Aki & Richards (2002), the positive Z-direction is downward, and as a result,
opposite sign of $r_1$ and $r_2$ represents retrograde particle motion. However, we note that in our analysis, the positive Z-direction is upward, and therefore, the same sign represents retrograde particle motion. It is also the convention for the synthetics in this paper. The different sign convention is also discussed in Aki & Richards (2002, Box 7.10 therein).

Using the program from Herrmann & Ammon (2002), we can calculate the eigenfunctions and ellipticity for different 1-D basin models. An example for a simple basin model with parameters described in Table 1 is shown in Figs 4(c) and (d). The basement depth of 4 km is comparable to the average depth in the study region. It is shallower than that beneath N116 in CVM-H model ($\sim$7 km in Fig. 1), but better fits the 4-s cut-off period of the higher mode.

From the eigenfunctions at 3-s period (Fig. 4c), we see that the particle motion at the surface differs in sign for the two modes. In addition, we see that the R-directed force can generate a stronger higher mode than Z-directed force, especially at greater depths, since $r_1$ decreases more slowly with depth than $r_2$ in 0–3 km depth range. The ellipticity over 1-10 s periods is shown in Fig. 4(d). In Supporting Information Fig. S3, we show similar plots for models with 6 and 8-km basement depths, and we see that distinct from the retrograde particle motion of the fundamental mode, the higher mode shows a prograde particle motion. The retrograde particle motion of higher mode only appears in the short period end (around 1-s period) in the 6 and 8-km depth basin models (Supporting Information Fig. S3). Therefore, the difference in the direction of particle motion, as is also evident in the data, can be useful to distinguish between the two modes in the 1–10 s period of interest.

In Fig. 4(d), we also note that at $\sim$4-s period ($T_0$), the higher mode disappears as shown by the truncation of the dispersion curve, which means no $k_1$ is found for the eigenproblem. It coincides with a rapid increase in the ellipticity of the fundamental mode (Fig. 4d), which is also shown in the observation (Fig. 4b). The cut-off period of the higher mode ($T_0$) can be estimated from the peak of the ellipticity of the fundamental mode ($T_1$), and both of them are controlled by the basement depth. This relationship is also shown in Supporting Information Fig. S3. However, in reality the structure is not 1-D, the cut-off period of the higher mode is more likely to be related to some average of the basement depth of the basin, while the fundamental-mode particle-motion ellipticity is related to the basement depth in the local site. For example, there is no such relationship between the fundamental and higher mode in Fig. 4(a), in which the receiver is at the Puente Hills. Nevertheless, using the receiver at the basin as in Fig. 4(b) gives a good estimate of $T_0$ from the fundamental mode.

The relative amplitude of the two modes is not only controlled by the structure between the two stations or the channels used in the cross-correlations (i.e. the Green’s function), but also the relative amplitude of the two modes generated by the noise sources (Snieder 2004). It is similar to the effect that an anisotropic distribution of noise sources causes asymmetry in the cross-correlation (e.g. Fig. 2a due to strong noise sources at the ocean side). We use the FK method (Zhu & Rivera 2002) to compute synthetics for the model in Table 1, with a point source (R- or Z-directed force) with central period of 3 s at 0.5 km depth. For a Z-directed source, the fundamental mode is dominant, even with a basin model.
Panels (a) and (b) show the particle motion measurements using the cross-correlations ($Z-Z$, $Z-R$, $R-R$, $R-Z$) between N116 and A135. N116 is located in the basin while A135 in the Puente Hills (Fig. 1). For each period, we have a maximum of four measurements corresponding to the two force directions at the virtual source and two modes. We only measure the particle motion when the phase is clear. The higher mode disappears above $\sim 4$-s period. The higher mode has a prograde particle motion, while the fundamental mode has a retrograde particle motion. A rapid increase in the ellipticity of the fundamental mode around $4$-s period is observed at N116. The observations can be explained with a 1-D basin model in Table 1. Panel (c) shows the eigenfunctions of this model, and panel (d) shows the phase velocity and particle-motion ellipticity ($H/V$) versus period. $T_0$: cut-off period of the higher mode; $T_1$: period having maximum fundamental-mode ellipticity.

Table 1. 1-D basin model.

<table>
<thead>
<tr>
<th></th>
<th>$H$ (km)</th>
<th>$V_S$ (km s$^{-1}$)</th>
<th>$V_P$ (km s$^{-1}$)</th>
<th>Rho (g cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sediments</td>
<td>4</td>
<td>1.0–3.0 (gradient)</td>
<td>2.5–5.0</td>
<td>2.0–2.5</td>
</tr>
<tr>
<td>Half space</td>
<td>3.8</td>
<td>6.5</td>
<td>2.8</td>
<td></td>
</tr>
</tbody>
</table>

Therefore, the Green’s function of a 1-D basin model cannot explain the strong higher mode we observe in the $Z-Z$ cross-correlations. Strong higher mode is generated by an $R$-directed source in a basin model (Supporting Information Fig. S4). Thus, we deduce that those noise sources that are effectively $R$-directed contribute to the strong higher mode in the cross-correlation. Considering the asymmetry of the cross-correlation and the 1–5 s period band discussed here, the primary noise sources are the secondary microseism (Longuet-Higgins 1950) that is equivalent to a vertical force (Gualtieri et al. 2013; Tanimoto et al. 2015). In contrast, the basin edge scattering has a significant horizontal component and can be important in generating the higher mode. The location of the edge of the LA Basin is evident from the basement depth map shown in Süss & Shaw (2003).

4 DISCUSSION

We see that with about 1-km interstation distance, we can extract strong surface wave signals as high as 1-Hz in frequency. The shorter period surface waves are sensitive to shallower depths, and therefore they are very useful to image the basin structure. Love wave signals are comparatively simple for dispersion analysis, but strong first higher-mode Rayleigh waves in the basin (e.g. Fig. 3) complicate the use of Rayleigh waves. The strong higher mode in the cross-correlation is not only due to the low-velocity sedimentary layer, but also the contribution from the $R$-directed noise sources, such as the basin edge scattering.

Rivet et al. (2015) recently proposed to use the $H/V$ ratio to distinguish between the modes. Their work involves inverting the dispersion curve for the $V_S$ model assuming separate cases as to whether the Rayleigh wave is a fundamental or higher mode. The $H/V$ ratios (over a range of periods) are then calculated for the two cases, and the one that best matches the data is chosen. In contrast, our proposed method distinguishes the two modes directly by measuring particle motion from the data.
In Fig. 5, we show the examples using the Z-Z and Z-R cross-correlations between N116 and A135. We plot the (unwrapped) instantaneous phase \( \phi(t) = \arg(r(t) + iz(t)) \), with a 5-point (0.5 s) moving-window smoothing. We then calculate the instantaneous curvature \( \phi''(t) \). The time \( t^* \), which corresponds to the maximum of \( \phi''(t) \), is where the two modes separate. The slope of \( \phi(t) \) should be negative for \( t < t^* \) (higher mode) and positive for \( t > t^* \) (fundamental mode) (Fig. 5a). If we only observe one mode, then we can deduce its type by the slope (Fig. 5b). This method can fail at high frequency when the two modes have the same sign. In Fig. 5(c), at 1-s period, the higher mode is only clear in the Z component, indicating a near-zero ellipticity, which is at the point where it is changing sign to retrograde (e.g. Supporting Information Fig. S3). In this case we can alternatively separate the modes based on the envelopes. In Fig. 6, we show the separations of the modes based on the method described here. Note that among all the cross-correlations in the profile, N116-A135 is the only one that belongs to the third case (Fig. 5c). The fundamental mode can also have a prograde particle motion at high frequency (Denolle et al. 2012).

Denolle et al. (2012) examined the particle motion direction of the fundamental mode at six stations in the LA basin surroundings, and found prograde direction in periods less than 2 s for the CHN station (see Fig. 1 for the location), where the sediment thickness is less than 1 km. The fundamental mode particle motion can be prograde if the velocity gradient in the sedimentary layer is very large (Tanimoto & Rivera 2005; Denolle et al. 2012). Nevertheless, a retrograde particle motion of the fundamental mode is a more general phenomena, as shown in the other 5 stations located in the deeper part of the LA Basin in Denolle et al. (2012) as well as our results above.

The Rayleigh-wave particle-motion ellipse has an observable inclination, which can be explained by body wave interference. As shown in Appendix B, the summation of one surface wave and one
body wave with arbitrary phase can be written as

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = M_{2 \times 2} \begin{bmatrix}
\cos(wt + \phi) \\
\sin(wt + \phi)
\end{bmatrix}
= U_{2 \times 2} \Sigma_{2 \times 2} V_{2 \times 2}^T \begin{bmatrix}
\cos(wt + \phi) \\
\sin(wt + \phi)
\end{bmatrix},
\] (2)

where \( M \) is related to the amplitudes and relative phase of the two waves; \( w \) is the angular frequency and \( wt + \phi \) is the phase of the surface wave (without interference); \( U, V \) and \( \Sigma \) are the matrices of singular value decomposition of the \( M \) matrix, where \( U \) and \( V \) are orthonormal matrices, and \( \Sigma \) is a diagonal matrix.

We recognize that the particle motion described by \( \begin{bmatrix}
\cos(wt + \phi) \\
\sin(wt + \phi)
\end{bmatrix} \) is a circle, \( V \) performs a rotation on the circle and the result remains a circle. The circle is then stretched to an ellipse with the operation of \( \Sigma \), and finally, the ellipse is rotated by the operator \( U \), which results in the tilted ellipse we observed. The surface and body waves, however, cannot be uniquely determined from the observed ellipse as we also show in Appendix B.

The cut-off period \( T_0 \), above which the first higher mode disappears, is controlled by the basement depth. In a 1-D model, this period coincides with the rapid increase in the fundamental mode ellipticity \((H/V)\) ratio and is close to the period \( T_1 = 1/f_1 \) with the peak \( H/V \) ratio (Fig. 4d). The frequency \( f_1 \) approximates the resonance frequency of the basin (Nakamura 1989; Field & Jacob 1993; Fäh et al. 2001; Boaga et al. 2013) and is related to the basement depth \( h \) by \( f_1 = V_S/2h \) from the constructive interference of shear waves. In reality the structure is not 1-D, but we can still use the fundamental mode ellipticity at the basin station to estimate \( T_0 \), with the assumption that the local basement depth is close to the average of the basin. For example, the data in Fig. 4(b) show the rapid change in the fundamental-mode ellipticity around 4-s period, which is about the \( T_0 \) shown there. It is also close to the \( T_1 \) estimated from a basement depth of 5 km (with \( V_S = 2 \text{ km s}^{-1} \)), which is the average basement depth in CVM-H model (Fig. 1). Alternatively, we can estimate the period in which the higher mode exists from the known basement depth measured from other methods, such as receiver functions.

5 CONCLUSIONS

We show some characteristics of the Love and Rayleigh waves in the Los Angeles Basin, which emerge in the multi-component cross-correlations using data from the LASSIE experiment. While the Love wave is simple, the Rayleigh wave is complicated due to the strong first higher mode. The low-velocity sedimentary layer and the radial-component noise sources together explain the strong first higher mode in the observations. We show that the particle motion direction can be used to distinguish between the two modes in the 1–10-s period of interest. The cut-off period of the first higher mode is controlled by the basement depth, which coincides with the rapid increase in the \( H/V \) ratio of the fundamental mode.

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REFERENCES


SUPPLEMENTARY INFORMATION

Additional Supporting Information may be found in the online version of this paper:

**Figure S1.** Love waves shown in the tangential-component cross-correlations between N116 and all the yellow stations in Fig. 1.
The cross-correlations filtered to 1 and 5-s period are shown as examples.

**Figure S2.** Rayleigh waves shown in the vertical and radial component cross-correlations \((Z-Z, Z-R, R-Z, \text{and } R-R)\) between N116 and all the yellow stations in Fig. 1. (a) Filtered to 1-s period. Both fundamental and higher modes are strong. (b) Filtered to 3-s period. The higher mode is generally stronger in the four profiles. (c) Filtered to 5-s period. Only the fundamental mode is observed.

**Figure S3.** The Rayleigh wave dispersion and ellipticity \((H/V)\) calculated for a basin model with a basement at: (a) 6-km depth; (b) 8-km depth. The other model parameters are same as that in Table 1.

**Figure S4.** The FK synthetics on the waveform in a 1-D basin model in Table 1. The source is at 0.5 km depth with a central period of 3 s. The waveforms are filtered to 3-s period. The top two panels show the \(Z\) and \(R\) recordings with a \(Z\)-directed source. The bottom two panels show the results for \(R\)-directed source.

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### APPENDIX A: \(H/V\) RATIO PRESERVE IN CROSS-CORRELATION

To enhance the extraction of surface wave signals, the data of each station are usually pre-processed before cross-correlation (Bensen et al. 2007; Lin et al. 2014). Here, we show that the Rayleigh wave that emerges in the cross-correlation preserves its \(H/V\) ratio, even with the pre-processing procedures (time domain normalization and spectral whitening) that modify the data.

In time domain normalization, we divide all three components \((Z/N/E)\) by the same trace, which is the smoothed envelope of the data filtered to the earthquake band (use the maximum of the three at each time point). Therefore, the arrival at a certain time \(t_i\) is scaled by the same factor \((m_i)\) at all three components. \(R\) component is the linear combination of \(N\) and \(E\), and therefore is also scaled by the same factor.

The Rayleigh wave in the cross-correlation is from the contribution of all the Rayleigh-wave arrivals from the noise sources at the stationary phase point. For illustration purpose, let us assume those \(N\) arrivals are spikes. For the \(R\)th arrival, let the amplitudes recorded at the \(Z/R\) component of the two stations be \(Z_{1i}/R_{1i}\) and \(Z_{2i}/R_{2i}\), and be scaled by \(m_1\) and \(m_2\) during the pre-processing. Then, the \(H/V\) ratio measured from \(Z-R\) and \(Z-Z\) cross-correlation is

\[
\frac{H}{V} = \frac{\sum_{i=1}^{N} \frac{1}{m_1 m_2} Z_{1i} \cdot R_{1i}}{\sum_{i=1}^{N} \frac{1}{m_1 m_2} Z_{1i} \cdot Z_{2i}}. \tag{A1}
\]

Rayleigh-wave \(H/V\) \((R/Z)\) ratio is a site property, which is controlled solely by the 1-D structure beneath the seismometer (Tanimoto & Rivera 2008; Yano et al. 2009). Let the \(H/V\) ratio at the second station be \(k_2\), then,

\[
\frac{R_{2i}}{Z_{2i}} = k_2 \tag{A2}
\]

\[
\frac{H}{V} = k_2 \frac{\sum_{i=1}^{N} \frac{1}{m_1 m_2} Z_{1i} \cdot R_{1i}}{\sum_{i=1}^{N} \frac{1}{m_1 m_2} Z_{1i} \cdot Z_{2i}} = k_2. \tag{A3}
\]

We see that the \(H/V\) ratio is preserved in the cross-correlation.

For the spectral whitening in frequency domain, we also do the same operation to all the three components, and the \(H/V\) ratio is also preserved.

### APPENDIX B: RAYLEIGH WAVE PARTICLE MOTION WITH BODY WAVE INTERFERENCE

The particle-motion ellipse of the Rayleigh wave is expected to be upright. However, it is observed to be tilted (Figs 4a and b). Here we show that the tilted ellipse can be explained by body wave interference. Let us consider a surface wave signal \((x(t), y(t))\) interfered by a body wave signal \((x_s(t), y_s(t))\) with phase lag \(\Delta \phi\):

\[
x(t) = x_s(t) + x(t) = A \cos(wt + \phi) + C \cos(wt + \phi + \Delta \phi)
y(t) = y_s(t) + y(t) = A \cos(wt + \phi \pm \pi/2) \cdot B + C \cos(wt + \phi + \Delta \phi) \cdot D.
\]

The result is

\[
x(t) = x_s(t) + x(t) = A \cos(wt + \phi) + C \cos(wt + \phi + \Delta \phi)
y(t) = y_s(t) + y(t) = AB \cos(wt + \phi) + CD \cos(wt + \phi + \Delta \phi - \sin(wt + \phi) \sin \Delta \phi).
\]

\[
x(t) = x_s(t) + x(t) = A \cos(wt + \phi) + C \cos(wt + \phi + \Delta \phi)
y(t) = y_s(t) + y(t) = CD \cos(wt + \phi) - CD \sin \Delta \phi \cdot AB \sin(wt + \phi).
\]
Let

\[ a = A + C \cos \Delta \phi \]
\[ b = -C \sin \Delta \phi \]
\[ c = CD \cos \Delta \phi \]
\[ d = -(CD \sin \Delta \phi \pm AB) \]

(B2)

and

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \]

we have

\[ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = M \begin{bmatrix} \cos(wt + \phi) \\ \sin(wt + \phi) \end{bmatrix}. \]

(B3)

In discrete time \((T_1, \ldots, T_N)\), eq. (B3) is

\[ \begin{bmatrix} X_1 & X_2 & \ldots & X_N \\ Y_1 & Y_2 & \ldots & Y_N \end{bmatrix}_{2 \times N} = M \begin{bmatrix} \cos(wT_1 + \phi) & \cos(wT_2 + \phi) & \ldots & \cos(wT_N + \phi) \\ \sin(wT_1 + \phi) & \sin(wT_2 + \phi) & \ldots & \sin(wT_N + \phi) \end{bmatrix}_{2 \times N}. \]

In short,

\[ \begin{bmatrix} X_{1 \times N} \\ Y_{1 \times N} \end{bmatrix} = M \begin{bmatrix} \cos(wT_{1 \times N} + \phi) \\ \sin(wT_{1 \times N} + \phi) \end{bmatrix}_{2 \times N}. \]

(B4)

With the singular value decomposition of \(M\), eq. (B4) is

\[ \begin{bmatrix} X_{1 \times N} \\ Y_{1 \times N} \end{bmatrix} = U_{2 \times 2} \Sigma_{2 \times 2} V_{2 \times 2}^T \begin{bmatrix} \cos(wT_{1 \times N} + \phi) \\ \sin(wT_{1 \times N} + \phi) \end{bmatrix}_{2 \times N}, \]

(B5)

where \(U\) and \(V\) are orthonormal matrices, and \(\Sigma\) is a diagonal matrix.

We recognize that the particle motion described by \(\begin{bmatrix} \cos(wT_{1 \times N} + \phi) \\ \sin(wT_{1 \times N} + \phi) \end{bmatrix}\) is a circle, \(V^T\) performs a rotation on the circle and the result remains a circle. The circle is then stretched to an ellipse with the operation of \(\Sigma\), and finally, the ellipse is rotated by the operator \(U\).

Therefore, we should expect to observe a tilted particle-motion ellipse of the surface wave, if it is interfered by the body wave.

Here we show an example.

Let \(A = 1, B = 2, \phi = 0, C = 0.2, \Delta = 0.5, D = \tan(\pi/3), \Delta \phi = \pi/9\).

The period \(T = 6\) s and \(w = 2\pi/T\). We plot one period, with sampling period \(dt = 0.01\) s. The result is shown below.

We proceed to show that the matrix \(M\) can be determined from the observation \(\begin{bmatrix} x^t \\ y^t \end{bmatrix}\).
Let $R = \begin{bmatrix} \cos(w T_{1 \times N} + \phi) \\ \sin(w T_{1 \times N} + \phi) \end{bmatrix}$

since

$$\int_0^T \cos^2 wtdt = \int_0^T \sin^2 wtdt = nT/2, \quad \int_0^T \cos wt \sin wt dt = 0,$$

where $T = 2\pi/w$ is the period, $n$ is the number of periods.

Then, if the observation evenly samples integer number of periods (with $N$ samples), we have

$$RR^T = \begin{bmatrix} N/2 \\ N/2 \end{bmatrix}.$$  \hfill (B7)

Recall that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = MR.$$

Then

$$\begin{bmatrix} X \\ Y \end{bmatrix} R^T = M RR^T = M \begin{bmatrix} N/2 \\ N/2 \end{bmatrix}.$$

$$M = 2/N \begin{bmatrix} X \\ Y \end{bmatrix}_{2 \times N} R^T$$

$$= 2/N \begin{bmatrix} X \\ Y \end{bmatrix}_{2 \times N} \begin{bmatrix} \cos(w T_{N \times 1} + \phi) & \sin(w T_{N \times 1} + \phi) \end{bmatrix}_{N \times 2}.$$  \hfill (B8)

The exact $\phi$ is unknown, because our phase measurement $\tilde{\phi}$ is based on ‘contaminated’ data. However, we can assume a weak body wave interference, and $\phi \approx \tilde{\phi}$.

Known $M$ thus $a$, $b$, $c$, $d$, we are unable to recover the surface wave and body wave signals uniquely since there exist five parameters $(A, B, C, D, \Delta\phi)$, unless we have constraint on at least one of them.