# Navier-Stokes Equations

The Navier-Stokes equations are the fundamental partial differentials equations that describe the flow of incompressible fluids. Using the rate of stress and rate of strain tensors, it can be shown that the components $F_j$ of a viscous force **F** in a nonrotating frame are given by

$$\frac{F_i}{V} = \frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\lambda\delta_{ij}\nabla\cdot\mathbf{u}\right] \quad (1)$$

$$= \frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)+\mu_B\delta_{ij}\nabla\cdot\mathbf{u}\right], \quad (2)$$

(Tritton 1988, Faber 1995), where $\eta$ is the dynamic viscosity, $\lambda$ is the second viscosity coefficient, $\delta_{ij}$ is the Kronecker delta, $\nabla\cdot\mathbf{u}$ is the divergence, $\mu_B$ is the bulk viscosity, and Einstein summation has been used to sum over $j$ = 1, 2, and 3.

Now, for an incompressible fluid, the divergence $\nabla\cdot\mathbf{u}=0$, so the $\lambda$ term drops out. Taking $\eta$ to be constant in space and writing the remainder of (2) in vector form then gives

$$\frac{\mathbf{F}_{\text{viscous}}}{V} = \eta\nabla^2\mathbf{u}, \quad (3)$$

where $\nabla^2\mathbf{u}$ is the vector Laplacian.

There are two additional forces acting on fluid parcels, namely the pressure force

$$\frac{\mathbf{F}_{\text{pressure}}}{V} = -\nabla P, \quad (4)$$

where $P$ is the pressure, and the so-called body force

$$\mathcal{F} = \frac{\mathbf{F}_{\text{body}}}{V}. \quad (5)$$

Adding the three forces (3), (4), and (5) and equating them to Newton's law for fluids yields the equation

$$\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P+\eta\nabla^2\mathbf{u}+\mathcal{F}, \quad (6)$$

and dividing through by the density $\rho$ gives

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho}+\nu\nabla^2\mathbf{u}+\frac{\mathcal{F}}{\rho}, \quad (7)$$

where the kinematic viscosity is defined by

$$\nu \equiv \frac{\eta}{\rho}. \quad (8)$$

The vector equations (7) are the (irrotational) Navier-Stokes equations.

When combined with the continuity equation of fluid flow, the Navier-Stokes equations yield four equations in four unknowns (namely the scalar $\rho$ and vector $\mathbf{u}$). However, except in degenerate cases in very simple geometries (such as Stokes flow, these equations cannot be solved exactly, so approximations are commonly made to allow the equations to be solved approximately.

As it must, the Navier-Stokes equations satisfy conservation of mass, momentum, and energy. Mass conservation is included implicitly through the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \Phi_{\text{mass}} = -\nabla \cdot (\rho \mathbf{u}) = -\mathbf{u} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{u}. \tag{9}$$

So, for an incompressible fluid,

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{u}. \tag{10}$$

Conservation of momentum requires

$$\frac{\text{momentum}}{\text{mass}} = [\text{mass}] + [\text{pressure}] + [\text{body force}] + [\text{viscosity}], \tag{11}$$

so

$$\frac{\partial \mathbf{u}}{\partial t} = (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla P + \frac{\mathcal{F}}{\rho} + \nu \nabla^2 \mathbf{u}. \tag{12}$$

Finally, conservation of energy follows from

$$\frac{\partial s}{\partial t} = -\mathbf{u} \cdot \nabla s + \frac{Q}{T}, \tag{13}$$

where $s$ is the entropy per unit mass, $Q$ is the heat transferred, and $T$ is the temperature.

Now consider the irrotational Navier-Stokes equations in particular coordinate systems. In Cartesian coordinates with the components of the velocity vector given by $\mathbf{u} = (u, v, w)$, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{14}$$

and the Navier-Stokes equations are given by

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x \tag{15}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y \tag{16}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z. \tag{17}$$

In cylindrical coordinates with the components of the velocity vector given by $\mathbf{u} = (u_r, u_\theta, u_z)$, the continuity equation is

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0, \tag{18}$$

and the Navier-Stokes equations are given by

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\phi^2}{r}\right)$$

$$=-\frac{\partial P}{\partial r}+\eta\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right)+F_r \qquad (19)$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_r u_\phi}{r}+\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi}+u_z\frac{\partial u_\phi}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \phi}+\eta\left(\frac{\partial^2 u_\phi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{\partial^2 u_\phi}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right)+F_\phi \qquad (20)$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi}+u_z\frac{\partial u_z}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial z}+\eta\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+F_z. \qquad (21)$$

In spherical coordinates with the components of the velocity vector given by $\mathbf{u}=(u_r,u_\theta,u_\phi)$, the continuity equation is

$$\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\theta\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}=0 \qquad (22)$$

and the Navier-Stokes equations are given by

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta^2}{r}-\frac{u_\phi^2}{r}\right)$$

$$=-\frac{\partial P}{\partial r}+\eta\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}-\frac{2u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_r}{\partial \theta}\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)+F_r \qquad (23)$$

$$\rho\left(\frac{\partial u_\theta}{\partial \theta}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_r u_\theta}{r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi^2\cot\theta}{r}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\eta\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)+F_\theta \qquad (24)$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_r u_\phi}{r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\theta u_\phi\cot\theta}{r}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)$$

$$=-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\eta\left(\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\phi}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right)+F_\phi. \qquad (25)$$

The Navier-Stokes equations with no body force (i.e., $\mathcal{F}=\mathbf{0}$)

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla P+\eta\nabla^2\mathbf{u} \qquad (26)$$

can be put into dimensionless form using the definitions

$$x' \equiv \frac{x}{L} \qquad (27)$$

$$y' \equiv \frac{y}{L} \qquad (28)$$

$$z' \equiv \frac{z}{L} \qquad (29)$$

$$\mathbf{u}' \equiv \frac{\mathbf{u}}{U} \qquad (30)$$

$$P' \qquad P$$

$$P' = \frac{-}{\rho U^2} \quad (31)$$

$$\nabla' \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x'} + \hat{\mathbf{y}}\frac{\partial}{\partial y'} + \hat{\mathbf{z}}\frac{\partial}{\partial z'} = L\nabla \quad (32)$$

$$t' \equiv t\frac{U}{L}. \quad (33)$$

Here, $U$ and $L$ are a characteristic velocity and a characteristic length. Then

$$\rho\frac{\partial(U\mathbf{u}')}{\partial\left(\frac{L}{U}t'\right)} + \rho U\mathbf{u}' \cdot \frac{1}{L}\nabla' U\mathbf{u}' = -\frac{1}{L}\nabla'(\rho U^2 P') + \eta\frac{1}{L^2}\nabla'^2(U\mathbf{u}'). \quad (34)$$

Assuming constant $\rho$ and multiplying both sides by $L/(\rho U^2)$ gives

$$\frac{\partial\mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla'\mathbf{u}' = -\nabla' P' + \frac{\eta}{L\rho u}\nabla'^2\mathbf{u}' \quad (35)$$

$$= -\nabla' P' + \frac{1}{\mathrm{Re}}\nabla'^2\mathbf{u}', \quad (36)$$

where Re is a dimensionless parameter known as the Reynolds number. Pressure is a parameter fixed by the observer, so it follows that the only other force is inertia force. Furthermore, the relative magnitudes of the pressure and inertial forces are describe by the Reynolds number, defined as

$$\mathrm{Re} = \frac{F_{\mathrm{inertia}}}{F_{\mathrm{viscous}}}. \quad (37)$$

For irrotational, incompressible flow with $\mathbf{F} = \mathbf{0}$, the Navier-Stokes equation then simplifies to

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u} \cdot \nabla\mathbf{u} = -\nabla P + \eta\nabla^2\mathbf{u}. \quad (38)$$

For low Reynolds number, the inertia term is smaller than the viscous term and can therefore be ignored, leaving the equation of creeping motion

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \nabla P = \eta\nabla^2\mathbf{u}. \quad (39)$$

In this regime, viscous interactions have an influence over large distances from an obstacle. For low Reynolds number flow at low pressure, the Navier-Stokes equation becomes a diffusion equation

$$\rho\frac{\partial\mathbf{u}}{\partial t} = \eta\nabla^2\mathbf{u}. \quad (40)$$

For high Reynolds number flow, the viscous force is small compared to the inertia force, so it can be neglected, leaving Euler's equation of inviscid motion

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u} \cdot \nabla\mathbf{u} = -\nabla P. \quad (41)$$

In the absence of a pressure force,

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u} \cdot \nabla\mathbf{u} = \eta\nabla^2\mathbf{u}, \quad (42)$$

which can be written as

$$\nabla \times \frac{\partial \mathbf{u}}{\partial t} - \nabla \times [\mathbf{u} \times (\nabla \times \mathbf{u})] = \nu \nabla^2 (\nabla \times \mathbf{u}). \quad (43)$$

For steady incompressible flow,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0}. \quad (44)$$

At low Reynolds number,

$$\nabla P = \eta \nabla^2 \mathbf{u}. \quad (45)$$

At low Reynolds number and low pressure

$$\nabla^2 \mathbf{u} = \mathbf{0}. \quad (46)$$

At high Reynolds number

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P. \quad (47)$$

For small pressure forces,

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = \eta \nabla^2 \mathbf{u}, \quad (48)$$

which can be written as

$$-\nabla \times [\mathbf{u} \times (\nabla \times \mathbf{u})] = \nu \nabla^2 (\nabla \times \mathbf{u}). \quad (49)$$

SEE ALSO: Continuity Equation, Euler's Equation of Inviscid Motion, Navier-Stokes Equations--Rotational, Reynolds Number, Stokes Flow, Stokes Flow--Cylinder, Stokes Flow--Sphere

## References


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