Seismic source spectra and moment tensors *

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When the source spectrum is expanded in powers of frequency, the coefficient of the \( n \)th power is a linear function of the \( n \)th moment tensor. This shows explicitly the low-frequency approximation by moments of low degree, and suggests suitable ways of extrapolation. In extrapolating to higher frequencies the spectral assumption implied restricts the source model to a particular class, and several possibilities, including \( \omega^2 \) and Gaussian models, are here considered. The model has 20 parameters, involving moments of degree zero, one, and two. Model-dependent constraints can reduce this number to that employed in other methods. Deterministic and stochastic interpretations are given, together with the relations to source dimension and correlation length. The models considered lead to simple expressions for the displacement field and the total radiated seismic energy, and can be made to satisfy certain general properties of observed far-field spectra, including the corner-frequency shift of P waves with respect to S waves. Moments of first degree determine the phase spectrum of the model, and their determination is equivalent to the classical source location problem by travel-time analysis. Moments of second degree control the spectral bandwidth and its variation with take-off angle and wave velocity, and their determination is comparable (but not equivalent) to corner-frequency methods of determining source dimensions. A preliminary investigation suggests that in particular the class of Gaussian models satisfies a number of criteria set forth in this paper.

A practical inversion method and its restrictions are discussed and illustrated by the analysis of SRO/ASRO data from a deep-focus event. It is observed that the digital SRO response presently employed can resolve only poorly the excitation spectra of this source, and that frequency-dependent \( Q \) and anomalous short-period amplitude variations can significantly affect the estimates. The numerical results in this example should be regarded as tentative.

1. Introduction

The quantification of seismic sources (e.g., by scalar moment or magnitude), the determination of source size (e.g., using corner-frequency methods), and the determination of radiation pattern (e.g., from fault-plane solutions) are procedures which usually proceed separately and use different parts of the seismic spectrum. Of course, the various source parameters are not independent of one another, and are often related through semi-empirical rules (Kanamori and Anderson, 1975). There would be obvious advantages if source parameters could be determined simultaneously.

The seismic-moment tensor represents both scalar moment and radiation pattern, but its applications are restricted to what may be regarded as point sources. In principle it is possible to remove this restriction by extending the representation to moment tensors of higher degree (Backus, 1977), and in practice it is possible to estimate moment tensors up to degree two (Doornbos, 1982). At this stage, however, a long-period approximation is still implied; it may be considered that moment tensors of degree up to two can summarise information from sources whose rise-time and spatial extent are smaller respectively than wave period and wavelength. In other words, spectral information above the corner frequency cannot be used in the present procedure.

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In this paper we discuss the possibility of using all of the spectral information to determine source parameters simultaneously, through the moment tensor approach. We will be guided by the consideration that, for practical purposes, the number of source parameters must be kept limited; more specifically, we retain the parameters involving moment tensors of degree up to two, as discussed in previous work (Doornbos, 1981, 1982). A spectral assumption is then needed to allow for the use of high-frequency information, although there should be no need to model the high-frequency part of individual source spectra precisely. In the following sections we discuss spectral models, including a class of $\omega^2$ and Gaussian models. It remains to test their consequences against observational evidence, but a preliminary investigation suggests that, in particular, the class of Gaussian models satisfies the criteria set forth here. We discuss the implications and limitations of this approach, and then illustrate the performance of a method based on such models by inverting long- and short-period data from the SRO network.

2. Moment Tensors and Low-Frequency Approximations

A representation by spatial and temporal moments requires the source function to be a transient in both space and time. The following modification of a conventional Green's function representation (e.g., Stump and Johnson, 1977; Aki and Richards, 1980, p. 59) is then useful:

$$u_i(x, t) = \int_{x_0} \int_{t_0} G'_{i,k}(\xi, x, t - \tau) m_{jk}(\xi, \tau) d\tau dV$$

where $u_i$ is displacement at position $x$ and at time $t$, $m_{jk}$ is the temporal derivative of the moment-tensor density, and $G'_{i,k}$ is a modified Green's function (Doornbos, 1982). In integral form,

$$G'_{i,k}(\xi, x, t - \tau) = \int_{\gamma_k/c} g'_{i,k}(\xi, x, t - \tau) d\rho$$

and, for asymptotic wave functions,

$$G'_{i,k}(\xi, x, t - \tau) = -\int_{\gamma_k/c} g'_{i,k}(\xi, x, t - \tau) d\rho$$

where $\rho$ is the ray parameter or slowness, $c$ the wave velocity, and $\gamma_k$ the direction cosine of the wave in $\xi$.

The expansion of $m_{jk}$ in moments of degree up to two requires 90 parameters, but if a common space and time history is assumed for the components of $m_{jk}$, i.e.,

$$m_{jk}(\xi, \tau) = M_{jk} f(\xi, \tau)$$

then the scalar function $f(\xi, \tau)$ can be expanded in moments and the representation up to second degree requires 20 parameters, six of which determine $M_{jk}$, the moment tensor of zero degree. The assumption (2) is introduced to reduce the number of source parameters for practical purposes, but the equation is satisfied for most of the "classical" source models (Haskell, 1964; Savage, 1966). The function $f(\xi, \tau)$ may be assumed normalised, and the expansion in moments relative to a reference point $(\xi_0, \tau_0)$ can be written

$$f(\xi, \tau) = (1 - F_i \partial_i - F_\tau \partial_\tau + \frac{1}{2} F_{im} \partial_i \partial_m + F_r \partial_r + \ldots) \cdot \delta(\xi - \xi_0) \delta(\tau - \tau_0)$$

where the summation convention is assumed for the indices $l$ and $m$, $F_i$ and $F_\tau$ are respectively the spatial and temporal components of $F_{(1)}$, the moment tensor of first degree. $F_{im}$ and $F_{ir}$ are the spatial—temporal components of $F_{(2)}$, the moment tensor of second degree. It may be noted that the $n$th-degree moment tensor of $f(\xi, \tau)$ is of order $n$, whereas the general $n$th-degree moment tensor of $m_{jk}(\xi, \tau)$ is of order $n + 2$:

$$M_{jk, im,...} = M_{jk} F_{im,...}$$

An interpretation of these moments in terms of kinematic source parameters has been given previously (Backus, 1977; Doornbos, 1981). The moments of first degree determine the "mislocation", i.e., the difference between the reference point $(\xi_0, \tau_0)$ and the source's "centre of gravity" $(\xi_0, \tau_0)$ for which $F_{(1)} = 0$ (called "the centroid" by Backus):

$$F_{(1)} = (\xi_0 - \xi_0, \tau_0 - \tau_0)$$

The moments of second degree are determined by the source's finiteness in space and time, and the rupture velocity. They also depend on the mislocation:

$$F_{(2)} = \hat{F}_{(2)} + F_{(1)} F_{(1)^T}$$
where \( \hat{F}(t) \) is measured with respect to \( (\hat{e}_0, \hat{e}_0) \). Only \( \hat{F}(t) \) represents intrinsic source parameters.

In the frequency–wavenumber domain, eq. (3) becomes

\[
F(\kappa, \omega) = (1 + i\kappa F_t - \omega F_r + \frac{1}{2} \kappa \kappa_m F_{m-t}) \times \exp[i(\kappa \cdot \xi_0 - \omega \tau_0)]
\]

(6)

which shows the moments as coefficients of a Taylor expansion about \( (\kappa, \omega) = (0, 0) \), and the first few terms given here form a low-frequency approximation of the spectrum. An integral representation of the displacement spectrum becomes

\[
U_1(x, \omega) = M_{jk} \int_{-\infty}^{+\infty} G'_{j,k}(\kappa, x, \omega) \times F(-\kappa, \omega) d\kappa / (2\pi)^3
\]

(7)

In many cases, the Green's function is strongly peaked in wavenumber space, and the integral in eq. (7) is trivial. For notational convenience, we drop the integral in most of what follows. We note that representing the Green's function in eq. (7) by the delta-function

\[
G'_{j,k}(\kappa, x, \omega) = G'_{j,k}(\xi, x, \omega) \times \exp[i(\kappa \cdot \xi_0)] \delta(\kappa - \kappa_0)
\]

would be equivalent to the Fraunhofer approximation often used in source analysis (Aki and Richards, 1980, p. 804).

If wavenumber components are interpreted in terms of propagating waves \( \kappa_t = \omega \gamma / c \), where \( c \) is the wave velocity and \( \gamma \) a direction cosine of the wave (possibly generalised in the sense of Richards, 1976). To be consistent with our notation and use of the Green's function, the wave direction defining \( \gamma \) is from the receiver to the source. Then a slowness vector can be formed:

\[
\hat{s} = (\gamma_1/c, \gamma_2/c, \gamma_3/c, 1)^T
\]

and an expression obtained for the displacement spectrum in terms of moment tensors:

\[
U_1(x, \omega) = M_{jk} G'_{j,k}(\xi_0, x, \omega) \left( 1 - i\omega \hat{s}^T F_{(1)} \right) - \frac{1}{2} \omega^2 \hat{s}^T F_{(2)} \hat{s} + \ldots \right) \exp(-i\omega \tau_0)
\]

(8)

Note that the spectrum of a one-dimensional function \( u(t) \) can be written in a similar form:

\[
U(\omega) = U(0) \left( 1 - i\omega U_{(1)} - \frac{1}{2} \omega^2 U_{(2)} + \ldots \right) \times \exp(-i\omega \tau_0)
\]

(9)

3. Source spectra with equivalent low-frequency approximations

From eqs. (5) and (8), the low-frequency excitation spectrum can be written as

\[
P(\hat{s}, \omega) = 1 - i\omega \hat{s}^T F_{(1)} - \frac{1}{2} \omega^2 \hat{s}^T \left( \hat{F}_{(2)} + F_{(1)} F_{(1)}^T \right) \hat{s}
\]

(10)

where it is recalled that the terms with \( F_{(1)} \) merely reflect the choice of reference point. The aim is to improve the source representation in the high-frequency range without increasing the number of parameters, while satisfying eq. (10) in the low-frequency limit. On the basis of these criteria we consider the following functions:

(a) a particular class of \( \omega^2 \) models (Aki, 1967):

\[
H(\hat{s}, \omega) = (1 - \frac{1}{2} \omega^2 \hat{s}^T \hat{F})^{-1} \exp(-i\omega \hat{s}^T F_{(1)})
\]

(11)

(b) a rational approximation, of the form

\[
K(\hat{s}, \omega) = \left[ \frac{1 - (1/4) \omega^2 \hat{s}^T \hat{F}_{(2)} \hat{s}}{1 + (1/4) \omega^2 \hat{s}^T \hat{F}_{(2)} \hat{s}} \right] \times \exp(-i\omega \hat{s}^T F_{(1)})
\]

(12)

(c) a class of Gaussian source spectra:

\[
N(\hat{s}, \omega) = \exp\left( -\frac{1}{2} \omega^2 \hat{s}^T \hat{F}_{(2)} \hat{s} - i\omega \hat{s}^T F_{(1)} \right)
\]

(13)

All of the models (11), (12) and (13) approach eq. (10) at low frequencies. Their relative performance at higher frequencies depends on the character of the actual source pulses, and the consequences should be tested against observational evidence; however, a preliminary investigation is conducted here on the basis of a few typical earthquake source pulses: a triangular pulse and a rectangular pulse. Of these, the triangular model seems to be more commonly used, sometimes in connection with a boxcar model. A common approximation to these two different functions should
tend to favour the class of spectral models which is sufficiently general to be applicable to a wide range of source functions.

The results are given in Figs. 1 and 2. It is shown that the low-degree moment-tensor approximation $P(\xi, \omega)$ is reasonable below the corner frequency, as are all the other equivalent low-frequency approximations. The rational function $K(\xi, \omega)$ is a good approximation up to at least twice the corner frequency, although its high-frequency behaviour illustrates that it does not really represent a physical source pulse. The spectral bandwidth of the particular $\omega^2$ model adopted is relatively large, as is the total energy, and a fit by this model would give second-degree moments which are correspondingly large. The Gaussian model leads to a reasonable approximation at both low and high frequencies, and the total energy is given to within $4\%$. It may be noted that on a logarithmic scale, the asymptotic Gaussian spectrum will be quite different from a conventional high-frequency asymptote (appropriate, for example, for an $\omega^2$ or $\omega^3$ model). However, for many purposes, including least-squares inversion, it is the absolute difference or difference squared that is important, and this difference goes to zero in the high-frequency limit.

4. Some implications

Having available a bounded source function, it is possible to compute the total radiated seismic energy. Equation (8) can be rewritten in the form

$$U_i(x, \omega) = M_{ij} G_j(x, \xi, \omega) F(\xi, \omega) \exp(-i \omega \tau_0)$$

(14)

There is no loss of generality in making use of the far-field Green’s function for a homogeneous medium having properties equivalent to those of
the region surrounding the source, and an expression for the seismic energy is then
\[
E_s = \frac{1}{16\pi^2\rho} \int_{0}^{\Omega} \left\{ \alpha^{-5} B^2(\xi_p) E(\xi_p) \left( \gamma_j \gamma_h M_{j,k} \right)^2 \\
+ \beta^{-2} B^2(\xi_s) E(\xi_s) \left[ \gamma_j \gamma_h M_{j,l} \right] \right\} \, d\Omega
\]
where \( \alpha \) and \( \beta \) are the P and S velocities, \( \xi_p \) and \( \xi_s \) are the associated slowness vectors, and we have introduced the energy \( E \) and spectral bandwidth \( B \) of a pulse according to
\[
E(\xi) = \int_{-\infty}^{\infty} \left( d\omega / 2\pi \right) |F(\xi, \omega)|^2
\]
\[
B^2(\xi) = \int_{-\infty}^{\infty} \left( d\omega / 2\pi \right) \omega^2 |F(\xi, \omega)|^2 / E(\xi)
\]
Thus the seismic energy depends only on two parameters of the source spectrum, and any spectral model which fits these parameters should give a good estimate, irrespective of the implied second-degree moments. For the models (11) and (13) of Section 3,
\[
B^2(\xi) E(\xi) = \chi \left( \xi^T \hat{F}_{(2)} \xi \right)^{-3/2}
\]
\[
\chi = \begin{cases} 1/4 & \text{(Gaussian)} \\ 1 & \text{ } \end{cases}
\]
(16)

In general it does not seem possible to evaluate analytically the integral over solid angle \( \Omega \) in eq. (15), except for the case of a point source. In the latter case eq. (15) reduces to an equivalent time-domain result of Rudnicki and Freund (1981), and \( \xi^T \hat{F}_{(2)} \xi \rightarrow \hat{\xi} \tau \). The result is
\[
E_s = \left( \chi \hat{\xi}^T \hat{F}_{(2)} \hat{\xi} / 60 \pi \rho \right) \left\{ \alpha^{-2} \left[ 2 M_{j,k} M_{j,k} + (M_{j,j})^2 \right] \\
+ \beta^{-2} \left[ 3 M_{j,k} M_{j,k} - (M_{j,j})^2 \right] \right\}
\]
However, it should be realised that, as pointed out previously (Backus, 1977), a point source in space but not in time is usually not a physically acceptable model, except perhaps for explosions. On the other hand, a point source in both space and time (i.e., with a step-function time dependence) implies that \( \hat{F}_{(2)} \rightarrow 0 \) and \( E_s \rightarrow \infty \), emphasising the fact that usually a point-source model can serve only as a low-frequency approximation. More generally, eqs. (15) and (16) imply that, for a fixed static moment, a smaller source radiates more seismic energy. Thus eq. (17) will usually overestimate the actual energy radiated.

The quadratic form in eq. (16) can be rewritten in the principal-axes system of \( \hat{F}_{(2)} \), the spatial part of \( \hat{F}_{(2)} \). Let \( p \) be the unit eigenvectors of \( \hat{F}_{(2)} \), and \( \lambda \) the associated (positive) eigenvalues. The spatial–temporal components \( \hat{F}_{(2)} \) are determined by the coupling between space and time through rupture velocity. Average rupture over the source, if nonzero, is expected to be in the direction of the major principal axis, say \( p_1 \). Then
\[
\xi^T \hat{F}_{(2)} \xi = (\lambda_1^2/c^2) (\gamma \cdot p_1)^2
\]
\[
+ (2/c) \hat{F}_{(2)} \gamma \cdot p_1 + \hat{F}_{(2)} \gamma \cdot p_1
\]
(18)
The reciprocal of this form determines the spectral bandwidth in all of the source models (10)–(13), even though the \( P(\xi, \omega) \) and \( K(\xi, \omega) \) models are invalid at high frequencies and only \( H(\xi, \omega) \) has a well-defined corner frequency. Since \( c \) is the P- or S-wave velocity, it is seen that the models usually predict a wider frequency band for P than for S, except in a range of directions for sources whose average rupture velocity is close to the S-wave velocity (unidirectional Haskell type of model). Note that the spectral bandwidth difference between P and S in these approximate models may arise irrespective of the properties of the actual source. For example, a Haskell-type model with bidirectional faulting is known to have a corner frequency for S that is usually higher than for P (Savage, 1972), whereas its moment tensor and associated low-frequency approximations (10)–(13) do not. It is therefore important that this particular spectral property of these models is consistent with what seems to be a consensus of observational opinion, namely, that corner frequencies for P are usually (though not always) higher than for S (Hanks, 1981).

The source models in Section 3 were introduced in the frequency–wavenumber domain, but it is also of interest to consider the interpretation in the space–time domain. In eqs. (11)–(13), the phase
factor involving \((\xi_0, \tau_0)\) cancels a similar factor in the displacement response (eq. (8)), showing merely that the result does not depend on the choice of reference point. The remaining source spectrum can be transformed to the space–time domain, and the Gaussian model gives the well-known expression for a normal distribution:

\[
N(\xi, \tau) = (2\pi)^{-1} [\hat{F}_{(2)}]^{-1/2} \times \exp \left[ -\frac{1}{2} \left( \frac{\xi - \hat{\xi}_0}{\tau - \hat{\tau}_0} \right)^T \hat{F}_{(2)}^{-1} \left( \frac{\xi - \hat{\xi}_0}{\tau - \hat{\tau}_0} \right) \right]
\]

(19)

Provided that this function is properly truncated, it can be interpreted deterministically in terms of the region occupied by the source in space and time. The spatial moments \(\hat{F}_{im}\) can be considered as representing a uniform distribution in space. Knowledge of the corresponding equivalent uniform source region would be useful for some purposes, including the determination of stress drop, although its value will underestimate the total region for a nonuniform source distribution. Similarly, the temporal moment \(\hat{F}_{\tau\tau}\) can be interpreted in terms of a uniform region in time. Thus, the equivalent uniform source region is

\[
V_u = (20/3)\pi\lambda_1\lambda_2\lambda_3
\]

\[
S_u = 4\pi\lambda_1\lambda_2
\]

\[
T_u = 2\sqrt{3} \hat{F}_{\tau\tau}^{1/2}
\]

(20)

where \(V_u\) is a volume, \(S_u\) is a surface appropriate for a plane fault, and \(T_u\) is a time length.

An alternative view is obtained by forming the frequency–wavenumber spectrum \(|\hat{F}(\xi, \omega)|^2\). For the Gaussian model, the transform to space and time gives

\[
|N(\xi, \omega)|^2 \rightarrow r(\xi, \tau) = (2\pi)^{-1} [\hat{F}_{(2)}]^{-1/2} \times \exp \left[ -\frac{1}{2} \left( \frac{\xi}{\tau} \right)^T \hat{F}_{(2)}^{-1} \left( \frac{\xi}{\tau} \right) \right]
\]

(21)

and this leads to correlation functions, which may be interpreted in a statistical sense. In this view, the source surface breaks coherently on a spatial and temporal scale given by the components of \(\hat{F}_{(2)}\), which may be small compared to the total source region. Such statistical source models have been discussed by Haskell (1966) and Aki (1967), although the assumed correlation functions were different; for example, Aki's \(\omega^2\) model assumes an exponential autocorrelation of the form \(\exp(-\alpha\tau)\).

It may be noted that a model described by Gaussian spatial correlation functions has been used, for similar reasons, to account for wave-scattering sources in the Earth (e.g., Chernov, 1960). In the Gaussian correlation model, scale lengths have been commonly identified with the \(e^{-1}\) correlation value. From eqs. (18) and (21), this would imply scale lengths of \(\hat{\xi}_1 = 2\lambda_1\), and \(\hat{\tau}_1 = 2\hat{F}_{\tau\tau}^{1/2}\), and the associated volume, surface and time length are

\[
V_c = (32/3)\pi\lambda_1\lambda_2\lambda_3
\]

\[
S_c = 4\pi\lambda_1\lambda_2
\]

\[
T_c = 4\hat{F}_{\tau\tau}^{1/2}
\]

(22)

5. Inversion techniques

In previous work, the time-domain version of eq. (8) was used as a basis for the inversion of long-period data (Doornbos, 1982):

\[
u_i(x, \tau) = M_{j\ell}G_{j\ell}(\xi_0, x, \tau - \tau_0) \star F(\xi, \tau)
\]

(23)

and for the Gaussian model, the source function \(F(\xi, \tau)\) becomes

\[
N(\xi, \tau) = (2\pi)^{-1/2} \times \exp \left[ -\left( \tau - \xi^T \hat{F}_{(2)} \xi \right)^2/2\xi^T \hat{F}_{(2)}^{-1} \xi \right]
\]

(24)

At this point it should be recalled that a complete representation requires integrating the forms (8) or (23) over the ray parameter or slowness. In an asymptotic approximation, the Green's function \(G_j\) is usually strongly peaked in slowness space and the integration is trivial, giving a number of terms representing for example P and S waves. In the moment-tensor approach (eq. (8)) the integral is not a real problem anyway, since the moment tensors are constants and the Green's functions and slowness vectors can be taken together to form new, modified Green's functions. With a Gaussian model in eq. (23) this is not possible. In the following we use data which are represented sufficiently by the asymptotic ap-
proximation. Should it happen that this is sufficient for the high-frequency part of a spectrum but not for the low-frequency part, then the two representations (23) and (8) can be used for alternative parts of the spectrum.

With these reservations, it is found that the moment tensors of the spectral model can be obtained in a way which is very similar to the moment-tensor inversion presented by Doornbos (1982). The linearised equations for perturbations of the moment-tensor components are

\[ u_j - \hat{u}_j = \left( \delta M_{jk} \mathcal{G}_{j,k} - M_{jk} \hat{\mathcal{G}}_{j,k} \right) \mathbb{I} \mathcal{T} \mathbf{F}_{(1)} + \frac{1}{2} M_{jk} \hat{\mathcal{G}}_{j,k} \mathbb{I} \mathcal{T} \mathbf{F}_{(2)} + N(\xi, \tau) \]

where \( \hat{u}_j \) is given by eq. (23), \( (M_{jk} + \delta M_{jk}) \) is taken to be an updated version of \( M_{jk} \), and similarly for the other moments. In practice, the solution should be constrained for it to be consistent with our conception of the mechanism of faulting.

It is often possible to treat the problem of determining the relocation vector \( \mathbf{F}_{(1)} \) separately. If Green's functions are determined by only one asymptotic wave, the form of eq. (24) shows that an estimate of

\[ \Delta T_i = \xi_i^T \mathbf{F}_{(1)} \]

can be obtained by standard travel-time analysis, and solving the linear system (26) would then amount to the usual procedure of estimating source location (e.g., Buland, 1976). In the following we assume \( \Delta T_i \) to be eliminated by "aligning" observed and synthetic seismograms.

Simple Green's functions in the above sense can be decomposed as

\[ \mathcal{G}_{j,k}(\xi_0, x, \omega) = (s_j \xi_k), G_j(\xi_0, x, \omega) \]

where \( s_j \) and \( \xi_k \) are components of the unit displacement vector and the slowness vector of the wave in \( \xi_0 \). This suggests a practical procedure for obtaining initial estimates of the moment tensors \( \mathbf{M} \) and \( \mathbf{F}_{(2)} \). Substituting the Gaussian model, eq. (8) can be rewritten as

\[ U_i(x, \omega) = A_i G_i(\xi_0, x, \omega) \exp\left(-\frac{1}{2} \omega^2 B_i + i \omega \tau_0 \right) \]

with

\[ A_i = (s_j \xi_k), M_{jk} \]

\[ B_i = \xi_i^T \mathbf{F}_{(2)} \xi_i \]

Observations in at least two frequency bands (e.g., long- and short-period data) are needed to estimate \( A_i \) and \( B_i \); then the linear systems (29) and (30) can be inverted. The second-degree moment tensor \( \mathbf{F}_{(2)} \) is determined by ten parameters, but constraints on the orientation of the source region and on the average rupture direction reduce this number to six. They are most easily obtained by means of eq. (18). Here the minor axis, say \( p_3 \), is inferred from the zero-degree moment tensor \( \mathbf{M} \) (apart from the so-called fault-plane ambiguity). One other parameter, \( \phi \), is then required to determine the orientations of \( p_1 \) and \( p_2 \). Thus, solutions for \( \lambda_1, \lambda_2, \lambda_3, F_{1r}, \) and \( F_{p-3} \) can be obtained as a function of \( \phi \). It should be noted, however, that a least-squares solution of the system (30) is actually determined by \( \log U_i \); it is therefore most sensitive to the very small amplitudes, where the effect of noise is relatively severe, and it is not a good measure of spectral bandwidth and energy. For these reasons it is suggested that the least-squares criterion be applied to the data set itself, and that the six parameters related to \( \mathbf{F}_{(2)} \) be estimated directly using eq. (28), or its equivalent time-domain expression.

6. An example

We have applied the technique outlined above in an inversion of long- and short-period data from the SRO/ASRO network. Partly to investigate its potential in applications with less sophisticated data, we here used information concerning body-wave polarity, long-period amplitudes, and short-period energy in the band 0.5—2.0 Hz. Amplitude data may be relevant for source pulses which are simple in the passband of the observations, and source parameters related to second-degree moments can be interpreted in terms of whole-source dimensions. If the source pulse is complicated, energy in one or more frequency bands should replace the amplitude information, and the interpretation of the second-degree moments is in terms of correlation distance and time. We compared long-period amplitude inversion with complete waveform inversion for the Bali Sea event used in previous work. The results as shown in
Table I are very similar, suggesting that at least for simple Green's functions, the data in a waveform are highly correlated, owing to their almost identical sampling of the focal sphere. The implications of the short-period data are discussed further at the end of this Section.

We have also applied the simultaneous long- and short-period inversion to a more recent event, for which more short-period data from the SRO/ASRO network were available. Figure 3 shows the vertical components of the body waves (including P, PKP, SKP) that were used; at present no horizontal-component short-period data are available. In the inversion the PREM velocity and density model (Dziewonski and Anderson, 1981) was used, together with the PREM Q-model for long-period data and the Q-model of Archambeau et al. (1969) for short-period data. Differences in Q between long- and short-period data have been suggested recently by several authors (e.g., Lundquist and Cormier, 1980). The inversion results are summarised in Table II and Fig. 4. Here the scalar moments $M$ and $N$ correspond to a (nonunique) decomposition of the assumed deviatoric tensor $M_{jk}$ into a major and a minor double couple; $M$ corresponds to the largest, $N$ to the smallest eigenvalue of $M_{jk}$. Standard deviations of the scalar moments were estimated by linearising the relationship with $M_{jk}$, whose standard deviations were obtained as a usual by-product of the linear least-squares procedure. Standard deviations of the second-degree moments were estimated by linearising the relationship between the solution and the data.

Apart from the zero-trace constraint on $M$ and the application of positivity constraints on the eigenvalues of $F_{(2)}$, no other constraints were applied in the inversion. Therefore the parameter values obtained do not necessarily correspond to a particular type of faulting. Also, the uncertainty in the results might be larger than the formal standard errors indicate. This is due partly to the rather small number of stations used and their poor azimuthal distribution. Figure 4 clearly shows that the nodal planes are poorly constrained by the polarities, and the nodal-plane solution is determined to a large extent by the relatively large PKP amplitudes in the N–E quadrant. The result should therefore be regarded as tentative, although it is not unusual in that it indicates down–dip compression (cf. Isacks and Molnar, 1971). Another source of uncertainty is the scatter in particular of the short-period data. Although we do not here investigate this question in detail, we speculate that systematic short-period amplitude anomalies may lead to source models having a relatively pronounced directivity pattern. For each phase, a Gaussian spectrum was fitted to the long- and short-period data after deconvolution of Earth and instrument responses. Figure 5 shows these spectra normalised to a unit DC level, and Fig. 6 gives the second moment ($B$, in eq. (28)), which is simply related to the spectral bandwidth and to the pulse width. The theoretical values for the source solution in Table II are also given in Fig. 6. The variation with take-off angle is determined by the spatial and spatial–temporal components of $F_{(2)}$, and eq. (18) provides insight concerning the maximum allowable variation. The quadratic form has a minimum for waves in the propagation direction of a unilateral propagating source (the Doppler effect), and it goes to zero only in the

### TABLE I

Comparison of amplitude inversion and waveform inversion for moment tensor of Bali Sea event of 1978, June 10 *

<table>
<thead>
<tr>
<th></th>
<th>$M_{11}$</th>
<th>$M_{22}$</th>
<th>$M_{33}$</th>
<th>$M_{12}$</th>
<th>$M_{13}$</th>
<th>$M_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>4.13</td>
<td>−1.77</td>
<td>−2.36</td>
<td>0.67</td>
<td>−0.48</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>(±0.51)</td>
<td>(±0.54)</td>
<td>(±0.90)</td>
<td>(±0.37)</td>
<td>(±0.27)</td>
<td>(±0.25)</td>
</tr>
<tr>
<td>Waveform</td>
<td>3.87</td>
<td>−0.77</td>
<td>−3.11</td>
<td>0.70</td>
<td>−0.58</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>(±0.88)</td>
<td>(±0.94)</td>
<td>(±1.56)</td>
<td>(±0.64)</td>
<td>(±0.46)</td>
<td>(±0.44)</td>
</tr>
</tbody>
</table>

* Standard deviations in parentheses. Data and other conditions as in case No. 4, tables 2 and 3 of Doornbos (1982).
Fig. 3. SRO and ASRO records from Fiji Islands event (see Table II for details) with vertical components of P, PKP or SKP. Note the different amplitude scales for the long- and short-period sections; the long-period record length is 4 min, the short-period record length, 12 s.

unlikely case of sonic rupture velocity. For sources with $\bar{F}_{rs} = 0$, the minimum is $\bar{F}_{rs}$ in the direction of the minor principal axis. On the other hand, the maximum is less than $\sim 5\bar{F}_{rr}$ for several fault models with subsonic rupture propagation. A final source of uncertainty is due to the fact that, for the data from this event, the sensitivity to source finiteness and also to take-off angle appears to be most important in the frequency range where the digital SRO response is small. The response curves in Fig. 5 serve to emphasise this fact. This may be a general problem with events in the magnitude range 5–6.

The total radiated seismic energy (Table II) was computed using eq. (15) or (17). We also inverted the data in terms of a point source in space but with a temporal moment $\bar{F}_{rr}$ (Table II, (b)). The corresponding seismic energy should be considered an upper bound. Finally, the assumption of a particular source model would predict spatial moments consistent with $\bar{F}_{rs}$, and the seismic energy for that model can be computed. For example, case (c) of Table II was obtained by approaching in the low-frequency limit the model of Molnar et al. (1973), for which spatial and temporal moments have been given previously (Doornbos, 1982). Thus, disregarding for the moment the possibility of systematic errors, it appears that seismic energy is estimated to within a factor of two.

In the context of a shear dislocation model, the apparent stress $\bar{n}\bar{\sigma}$ may be obtained from the scalar moment $M$ and the seismic energy $E_s$, and the stress drop $\Delta\sigma$ may be obtained from the scalar moment and the fault shape and surface area (e.g., Aki, 1972). The fault surface area in Table II was computed using eq. (20), and the stress drop was
| $M$ (10$^{25}$ dyne cm) | $N$ (10$^{25}$ dyne cm) | $\lambda_1^2$ (km$^2$) | $\lambda_2^2$ (km$^2$) | $\lambda_3^2$ (km$^2$) | $|\vec{E}_r|$ (km s) | $\vec{E}_T$ (s$^2$) | RMS error | Other parameters |
|-------------------------|-------------------------|----------------------|----------------------|----------------------|-------------------|------------------|-----------|-----------------|
| (a) 1.92 ($\pm 0.06$)   | 0.17 ($\pm 0.05$)       | 16.8 ($\pm 0.9$)     | 15.7 ($\pm 1.3$)     | 1.5 ($\pm 3.5$)      | 2.0 ($\pm 0.1$)   | 0.45 ($\pm 0.06$) | 44        | $E_r = 2.4 \times 10^{19}$ erg |
|                         |                         |                      |                      |                      |                   |                  |           | $S = 204 \text{ km}^2$ |
|                         |                         |                      |                      |                      |                   |                  |           | $\Delta \sigma = 16 \text{ bar}$ |
|                         |                         |                      |                      |                      |                   |                  |           | $\eta \sigma = 1.5 \text{ bar}$ |
| (b) 1.92 ($\pm 0.06$)   | 0.17 ($\pm 0.05$)       | 0 ($\pm 0.05$)       | 0 ($\pm 0.05$)       | 0 ($\pm 0.05$)       | 0 ($\pm 0.05$)    | 0.33 ($\pm 0.02$) | 148       | $E_r = 4.4 \times 10^{19}$ erg |
|                         |                         |                      |                      |                      |                   |                  |           | $S = 118 \text{ km}^2$ |
|                         |                         |                      |                      |                      |                   |                  |           | $\Delta \sigma = 36 \text{ bar}$ |
|                         |                         |                      |                      |                      |                   |                  |           | $\eta \sigma = 2.0 \text{ bar}$ |
| (c) 1.92 ($\pm 0.06$)   | 0.17 ($\pm 0.05$)       | 9.4 ($\pm 1.2$)      | 9.4 ($\pm 1.2$)      | 0 ($\pm 0.02$)       | 0 ($\pm 0.02$)    | 0.25 ($\pm 0.02$) | 155       | $E_r = 3.2 \times 10^{19}$ erg |

* $M$ and $N$ are the scalar moments of the major and the minor double couples, $\lambda_1^2$ are the positive eigenvalues of the spatial moment tensor, $\vec{E}_r$ is the temporal moment, $|\vec{E}_T|$ is the length of the spatial-temporal moment, $E_r$ is the total radiated seismic energy, $S$ is fault surface area, $\Delta \sigma$ is stress drop, $\eta \sigma$ is apparent stress. Standard deviations in parentheses. Results for (a) general model, (b) point source, (c) prescribed rupture on circular fault.
Fig. 4. Fault-plane solution for major double couple, with observed polarities and reduced long-period amplitudes of P(±) and SH(→). P and T are the compression and tension axes; \( p_1, p_2, p_3 \) are the principal axes of the source ellipsoid (in decreasing order).

Calculated simply for a circular fault. Despite the rather different constraints the results for cases (a) and (c) are reasonably close, suggesting that useful estimates of seismic energy and stress drop can be obtained without knowing the fault geometry and rupture history in detail. The estimated fault surface area seems large compared to estimates for similar earthquakes by Chung and Kanamori (1980), but small compared to the results of Wyss and Molnar (1972). Consequently, the estimated stress drop also lies between the results of these authors.

We now consider the implications of the short-period data in connection with the Q-model being used. We repeated the inversion with short-period energy in the frequency bands 0–1.0 and 0–0.5

Fig. 5. Gaussian spectra fitted to the long- and short-period data of Fig. 3, after deconvolution of Earth and instrument responses. The digital long- (Lp) and short-period (Sp) SRO responses are also indicated (different scales).
Fig. 6. Second moment of Gaussian source spectrum \(\gamma_i\) in eq. (26): \(\bigcirc\), observed; \(\bigcirc\), excitation for source model in Table II. The station sequence is as in Fig. 3.

Hz, respectively. For a spatial point source this gave \(\hat{F}_{rr} = 0.82 \text{ s}^2\) and \(1.47 \text{ s}^2\), as compared to \(0.33 \text{ s}^2\) for the frequency band \(0.5-2.0 \text{ Hz}\). These results are illustrated by the excitation spectra in Fig. 7. This figure also shows the spectrum of a typical P-wave Green’s function at teleseismic distance in a standard Earth model and at the output of the digital short-period SRO system. Using the \(Q\)-model of Archambeau et al. (1969) which was used to deconvolve the short-period data, the peak is near 1.4 Hz. But using the PREM \(Q\)-model applied in the deconvolution of the long-period data, the peak is near 0.8 Hz. The conclusion is that a frequency-independent \(Q\)-model affects the estimate of source pulse length; it similarly affects estimates of source size, seismic energy and related quantities. This corroborates a conclusion reached by Choy and Boatwright (1981), also in a source analysis study. Der (1981) rather extensively documented the present discrepancy between current \(Q\)-models and high-frequency data, and Lundquist and Cormier (1980) have made an attempt to estimate the frequency dependence of \(Q\). Quantitative source analysis can be expected to improve significantly if adequate corrections can be obtained for this effect, and for the effect of anomalous amplitude variations.

Fig. 7. Gaussian excitation spectra for spatial point source with finite temporal moment obtained by fitting long-period amplitudes and short-period energy: (1) short-period bandwidth 0.5-2.0 Hz, temporal moment \(\hat{F}_{rr} = 0.33 \text{ s}^2\); (2) bandwidth 0-1.0 Hz, \(\hat{F}_{rr} = 0.82 \text{ s}^2\); (3) bandwidth 0-0.5 Hz, \(\hat{F}_{rr} = 1.47 \text{ s}^2\). Curve (a) indicates the digital short-period SRO response. Curves (b) and (c) indicate a typical Green’s function in a standard Earth model, for P at teleseismic distance: (b) PREM \(Q\)-model (used with the long-period data); (c) \(Q\)-model from Archambeau et al. (used with the short-period data). Note the different amplitude scales for the different functions.

7. Conclusion

The general moment-tensor representation of seismic sources becomes a low-frequency approximation when only the moments of lower degree are retained. A complete representation above the corner frequency requires moment tensors of degree higher than two, but a useful approximation is given by the low-degree moments together with a spectral assumption; such an assumption restricts the source models to a particular class. These models can be interpreted in a deterministic or a stochastic sense, they lead to simple expressions for the displacement field and the total radiated seismic energy, and they can be made to satisfy certain general properties of observed far-field spectra, including the corner-frequency shift for P waves with respect to S waves. It remains to test the results of these models against observational evidence, but a preliminary investigation using synthetic waveforms suggests that, of the models considered, the class of Gaussian models in particular satisfies the criteria set forth in this paper. Although on a
logarithmic scale the high-frequency part of the Gaussian spectrum may differ significantly from conventional \( \omega^2 \) or \( \omega^3 \) asymptotes, the absolute amplitude difference is small and would not significantly affect least-squares estimates of the source parameters.

A mislocation of the source gives rise to moments of first degree. They determine the phase spectrum in the spectral model, and can often be analysed separately by standard travel-time analysis. The source's spatial and temporal extent give rise to moments of second degree, which control the spectral bandwidth and its variation with take-off angle and wave velocity. In the past, much emphasis has been placed on the corner frequency as a measure of the spectral bandwidth and as a means to determine the source dimension. Brune's (1970) model is often used, partly because it provides an estimate of source dimension even from a single station record. It is possible to do this in the context of a moment-tensor spectral model too: if source shape and rupture velocity are prescribed, the number of independent second-degree moments is two; if the fault is circular, this number reduces to one.

Long- and short-period data, or more precisely, data in at least two frequency bands, can be inverted by a perturbation method analogous to that used in previous work. However, it is often possible to obtain initial estimates of the moments of degree zero and two separately, where the zero-degree moments are obtained by linear inversion, and the second-degree moments by nonlinear inversion. In an application of this technique to the analysis of a deep event in the Fiji Islands region we observed that (1) the obtainable resolution of the spatial and temporal parameters of this source would be maximum in a frequency range where the digital SRO response is small, (2) knowledge of the frequency dependence of \( Q \) is needed to reconcile long- and short-period data, or even data within the short-period band, and (3) amplitude variations are sometimes larger than explained by source radiation and standard propagation effects. Points (2) and (3) corroborate the results of other workers. The data set used in our example was rather limited, consisting of the polarity and long- and short-period amplitudes of ten body-wave phases. Since no attempt was made to correct for the above-mentioned effects, the numerical results obtained should be regarded as tentative.

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