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FINAL STAGES OF PLANET FORMATION

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ABSTRACT

We address three questions regarding solar system planets: What determined their number? Why are their orbits nearly circular and coplanar? How long did they take to form?

Runaway accretion in a disk of small bodies resulted in a tiny fraction of the bodies growing much larger than all the others. These big bodies dominated the viscous stirring of all bodies. Dynamical friction by small bodies cooled the random velocities of the big ones. Random velocities of small bodies were cooled by mutual collisions and/or gas drag. Runaway accretion terminated when the orbital separations of the big bodies became as wide as their feeding zones. This was followed by oligarchic growth during which the big bodies maintained similar masses and uniformly spaced semimajor axes. As the oligarchs grew, their number density decreased, but their surface mass density increased. We depart from standard treatments of planet formation by assuming that as the big bodies got bigger, the small ones got smaller as the result of undergoing a collisional fragmentation cascade. It follows that oligarchy was a brief stage in solar system evolution.

When the oligarchs' surface mass density matched that of the small bodies, dynamical friction was no longer able to balance viscous stirring, so their velocity dispersion increased to the extent that their orbits crossed. This marked the end of oligarchy. What happened next differed in the inner and outer parts of the planetary system. In the inner part, where the ratios of the escape velocities from the surfaces of the planets to the escape velocities from their orbits are smaller than unity, big bodies collided and coalesced after their random velocities became comparable to their escape velocities. In the outer part, where these ratios are larger than unity, the random velocities of some of the big bodies continued to rise until they were ejected. In both parts, the number density of the big bodies eventually decreased to the extent that gravitational interactions among them no longer produced large-scale chaos. After that their orbital eccentricities and inclinations were damped by dynamical friction from the remaining small bodies.

The last and longest stage in planet formation was the cleanup of small bodies. Our understanding of this stage is fraught with uncertainty. The surviving protoplanets cleared wide gaps around their orbits that inhibited their ability to accrete small bodies. Nevertheless, in the inner planet system, all of the material in the small bodies ended up inside planets. Small bodies in the outer planet system probably could not have been accreted in the age of the solar system. A second generation of planetesimals may have formed in the disk of small bodies, by either collisional coagulation or gravitational instability. In the outer planet system, bodies of kilometer size or larger would have had their random velocities excited until their orbits crossed those of neighboring protoplanets. Ultimately they would have either escaped from the Sun or become residents of the Oort Cloud. An important distinction is that growth of the inner planets continued through cleanup, whereas assembly of the outer planets was essentially complete by the end of oligarchy. These conclusions imply that the surface density of the protoplanetary disk was that of the minimum solar mass nebula in the inner planet region but a few times larger in the outer planet region. The timescale through cleanup was set by the accretion rate at the geometrical cross section in the inner planet region and by the ejection rate at the gravitationally enhanced cross section in the outer planet region. It was a few hundred million years in the former and a few billion years in the latter. However, since Uranus and Neptune acquired most of their mass by the end of oligarchy, they may have formed before Earth!

A few implications of the above scenario are worth noting. Impacts among protoplanets of comparable size were common in the inner planet system but not in the outer. Ejections from the outer planet system included several bodies with masses in excess of Earth after oligarchy and an adequate number of kilometer-size bodies to populate the Oort comet cloud during cleanup. Except at the very end of cleanup, collisions prevented Uranus and Neptune from ejecting kilometer-size objects. Only Jupiter and, to a much lesser extent, Saturn were capable of populating the Oort Cloud with comets of kilometer size.

Subject headings: planetary systems: protoplanetary disks — solar system: formation

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1. INTRODUCTION

Modern scenarios for planet formation may be broken down into several stages. The growth of the smallest gravitationally active bodies, planetesimals, is mired in controversy (Lissauer 1993; Youdin & Shu 2002). Orderly growth by the merging of planetesimals is followed by runaway accretion in which a small fraction of the bodies grow much larger than all the others (Safronov 1972; Wetherill & Stewart 1989). When these big bodies are sparse enough so that each dominates viscous stirring in its feeding zone, runaway growth gives way to oligarchic growth during which the big bodies grow in lockstep, maintaining similar masses and uniformly spaced orbits (Kokubo & Ida 1998). As oligarchs grow, their orbital spacing increases and their number decreases. We investigate how oligarchy ends and what happens after it does. The plan of our paper is as follows. We describe the conditions that pertain at the end of oligarchy in § 2. We show in § 3 that at this stage dynamical friction from the small bodies is no longer able to balance the mutual stirring of the big bodies. In § 4 we treat the regularization of the orbits of the big bodies and the cleanup of small bodies. We summarize our findings in § 5.1.

A few definitions are in order. For simplicity, we consider two classes of bodies, big ones and small ones, each composed of material density ρ. We denote the surface mass density, random velocity dispersion, and radius of the former by Σ, v, and R and of the latter by σ, u, and s. The distance from and angular velocity about the Sun are given by a and Ω. We refer to the regions of the terrestrial and ice-giant planets as, respectively, the inner and outer planet systems. We do not call the latter the outer solar system, so as not to confuse it with the Oort Cloud. In our numerical estimates, we set $\rho = 5.5$ g cm^{-3}, the density of Earth, at 1 AU and $\rho = 1.5$ g cm^{-3}, approximately the densities of Uranus and Neptune, at 25 AU. For the condensable fraction of the protoplanetary nebula, we adopt the surface densities $\sigma = 7$ g cm^{-2} at 1 AU and $\sigma = 1.5$ g cm^{-2} at 25 AU. The former is just that appropriate to the minimum mass solar nebula (Hayashi 1981), but the latter is enhanced sixfold relative to it. This enhancement is designed to make the isolation mass in the outer planet system comparable to the masses of Uranus and Neptune. This is necessary, since the timescale for the accumulation of the outer planets by coagulation of smaller isolation masses would exceed the age of the solar system. The particular value of 6 applies if just half the mass had accreted into protoplanets by the end of oligarchy. Instead, if most of it had, 3 would be the appropriate enhancement factor. In evaluating expressions containing the planet mass, M_p, we use 1 $M_\oplus$ for an inner planet and 15 $M_\oplus$ for an outer one, where $M_\oplus \approx 6.0\times10^{27}$ g is Earth's mass.

It proves convenient to employ a symbol α for the ratio between the radius of a body and that of its Hill sphere, $R_{\rm H}$:

$$\alpha \equiv \left(\frac{9}{4\pi}\frac{M_\odot}{\rho a^3}\right)^{1/3} \approx \begin{cases} 234^{-1}, & a = 1 \text{ AU}, \\ 3800^{-1}, & a = 25 \text{ AU}. \end{cases} \quad (1)$$

Note that α is approximately the angle subtended by the Sun at a. We also make use of the Hill velocity of the big bodies, $v_{\rm H} \sim \Omega R_{\rm H} \sim \alpha^{1/2} v_{\rm esc}$, where $v_{\rm esc}$ is the escape speed from the surface of a big body.

We offer an interpretation of how the basic properties of the Sun's planetary system might have been established. Our emphasis is on the poorly explored stages that follow oligarchy. We are more concerned with proof of principal than with examining different scenarios. Thus, except for a few remarks in § 5.2, we neglect the effects of gas drag. We argue that as big bodies grow and viscous stirring intensifies, small bodies are collisionally fragmented. Fragmentation lowers their random velocities, allowing them to be accreted more easily.

2. OLIGARCHY UNTIL ISOLATION, $\Sigma < \sigma$

Protoplanet growth is oligarchic until $\Sigma \approx \sigma$, which we refer to as the epoch of isolation. Oligarchy comes in two different flavors, shear dominated and dispersion dominated. These are discussed at length in Goldreich et al. (2004). Below we list some highlights.

2.1. *Shear-dominated Oligarchy, $u < v_{\rm H}$*

In this regime the big bodies heat each other and are cooled by dynamical friction from the small bodies at the rates[4]

$$\frac{1}{v}\frac{dv}{dt} \sim \frac{\Sigma\Omega}{\rho R}\alpha^{-2}\frac{v_{\rm H}}{v} - \frac{\sigma\Omega}{\rho R}\alpha^{-2}. \quad (2)$$

At equilibrium, $dv/dt = 0$,

$$v \sim \frac{\Sigma}{\sigma} v_{\rm H}. \quad (3)$$

This justifies the use of the heating rate appropriate to $v < v_{\rm H}$ in equation (2).

Oligarchs' orbits maintain separations of the order of the widths of their feeding zones $\sim 5R_{\rm H}$. As they grow, their feeding zones overlap, and neighboring oligarchs may coalesce. Thus, the masses of the oligarchs increase by merging, as well as by accretion of small bodies. The ratio of the growth rate by the former process to that by the latter is

$$\frac{(\Sigma\Omega/\rho R)\alpha^{-3/2}}{(\sigma\Omega/\rho R)\alpha^{-1}(v_{\rm H}/u)} = \frac{\Sigma}{\sigma}\alpha^{-1/2}\frac{u}{v_{\rm H}}, \quad (4)$$

after neighboring oligarchs occupy a common feeding zone. At the end of oligarchy, when $\Sigma \sim \sigma$, merging limits the number of oligarchs in a given feeding zone to a small value provided that $u > \alpha^{1/2} v_{\rm H}$.

2.2. *Dispersion-dominated Oligarchy, $u > v_{\rm H}$*

Big bodies heat each other and are cooled by dynamical friction from the small bodies at the rates

$$\frac{1}{v}\frac{dv}{dt} \sim \frac{\Sigma\Omega}{\rho R}\alpha^{-2}\left(\frac{v_{\rm H}}{v}\right)^4 - \frac{\sigma\Omega}{\rho R}\alpha^{-2}\left(\frac{v_{\rm H}}{u}\right)^4. \quad (5)$$

Equilibrium occurs at

$$\frac{v}{u} \sim \left(\frac{\Sigma}{\sigma}\right)^{1/4}. \quad (6)$$

This justifies the use of the heating rate appropriate to $v > v_{\rm H}$ near the end of oligarchy, when Σ is only slightly smaller than σ.

[4] The heating rate given in the literature is proportional to $(v_{\rm H}/v)^2$ instead of to $v_{\rm H}/v$. Goldreich et al. (2004) show that the latter is correct.

What happens when the feeding zones of neighboring oligarchs overlap? The ratio of the merging rate to the growth rate by accreting small bodies is

$$\frac{(\Sigma\Omega/\rho R)\alpha^{-1}(v_{\rm H}/v)^2}{(\sigma\Omega/\rho R)\alpha^{-1}(v_{\rm H}/u)^2} = \left(\frac{\Sigma}{\sigma}\right)^{1/2}. \quad (7)$$

Thus, prior to isolation, mergers are rare, and there are many large bodies within a common feeding zone. These large bodies experience runaway growth relative to one another; only the largest grows appreciably.

2.3. *Conditions at Isolation for $u \approx v_{\rm H}$*

Sections 2.1 and 2.2 show that oligarchy proceeds in either the shear-dominated or dispersion-dominated regime provided that $\Sigma < \sigma$. We demonstrate below that oligarchy ends when $\Sigma \sim \sigma$, the epoch of isolation. Here, for simplicity, we evaluate all quantities at isolation under the assumption that u is fixed at the boundary between the shear- and dispersion-dominated regimes, that is, $u/v_{\rm H} \sim 1$. Generalization to other values of $u/v_{\rm H}$ is given in § 2.4 along with the dependence of $u/v_{\rm H}$ on s.

With $u = v_{\rm H}$ throughout oligarchy, $v \leq u$ with equality obtaining at isolation (see eqs. [3] and [6]). Thus, each oligarch accretes material from within an annulus of half-width $2.5R_{\rm H}$ (Petit & Henon 1986), and the isolation radius

$$R_{\rm iso} \approx \left(\frac{15}{4}\frac{\sigma a}{\rho\alpha}\right)^{1/2} \approx \begin{cases} 1.3\times10^3 \text{ km}, & a = 1 \text{ AU}, \\ 2.3\times10^4 \text{ km}, & a = 25 \text{ AU}. \end{cases} \quad (8)$$

Equivalently, the isolation mass

$$M_{\rm iso} \approx \frac{4\pi}{3}\rho R_{\rm iso}^3 \approx \begin{cases} 8\times10^{-3}\ M_\oplus, & a = 1 \text{ AU}, \\ 1.3\times10^1\ M_\oplus, & a = 25 \text{ AU}. \end{cases} \quad (9)$$

The number of oligarchs per unit logarithmic semimajor axis is

$$N_{\rm iso} = \frac{\pi\sigma a^2}{M_{\rm iso}} \approx \begin{cases} 100, & a = 1 \text{ AU}, \\ 9, & a = 25 \text{ AU}. \end{cases} \quad (10)$$

Equations (8)–(10) apply for $u < v_{\rm H}$. However, from here to the end of § 2.3 we specialize to $u = v_{\rm H}$.

With $u = v_{\rm H}$, the ratio of the accretion cross section to the geometric one is $(v_{\rm esc}/u)^2 \approx \alpha^{-1}$. Thus, the timescale from the start of oligarchic growth until isolation is

$$t_{\rm iso} \sim \Omega^{-1}\alpha^{1/2}\left(\frac{\rho a}{\sigma}\right)^{1/2} \sim \begin{cases} 10^5 \text{ yr}, & a = 1 \text{ AU}, \\ 10^7 \text{ yr}, & a = 25 \text{ AU} \end{cases} \quad (11)$$

(see, e.g., Lissauer 1987).

At isolation, the escape velocity from the surface of an oligarch is given by

$$v_{\rm esc} = (2GM_{\rm iso}/R_{\rm iso})^{1/2} \approx \begin{cases} 2.3 \text{ km s}^{-1}, & a = 1 \text{ AU}, \\ 21 \text{ km s}^{-1}, & a = 25 \text{ AU}. \end{cases} \quad (12)$$

These values are to be compared with the escape velocity from solar orbit, 44 km s^{-1} at 1 AU and 8.8 km s^{-1} at 25 AU.

Viscous stirring of small bodies by oligarchs at isolation results in collisions at speeds $u_{\rm col} \sim v_{\rm H}$, so

$$u_{\rm col} \sim \alpha^{-1}R_{\rm iso}\Omega \approx \begin{cases} 60 \text{ m s}^{-1}, & a = 1 \text{ AU}, \\ 140 \text{ m s}^{-1}, & a = 25 \text{ AU}. \end{cases} \quad (13)$$

2.4. *Sizes and Velocities of Small Bodies*

The size, s, and velocity dispersion, u, of the small bodies are uncertain. They are also closely related: u is set by an equilibrium between viscous stirring by the big bodies and damping by collisions between small bodies, which occur at a rate that is inversely proportional to s. To maintain $u/v_{\rm H} \sim 1$ at isolation requires the effective radius of the small bodies to take on the particular value

$$s_b \sim \alpha^{3/2}\left(\frac{\sigma a}{\rho}\right)^{1/2} \sim \begin{cases} 10 \text{ m}, & a = 1 \text{ AU}, \\ 1 \text{ m}, & a = 25 \text{ AU}. \end{cases} \quad (14)$$

In the shear-dominated regime, $s < s_b$, $u/v_{\rm H} \sim s/s_b$. This does not affect the $2.5R_{\rm H}$ half-width of an oligarch's feeding zone. Hence, $R_{\rm iso}$, $M_{\rm iso}$, and $N_{\rm iso}$ are still given by equations (8), (9), and (10). Moreover, $u_{\rm col}$ remains comparable to $v_{\rm H}$ (eq. [13]), the typical random velocity at which a small body exits an oligarch's Hill sphere. However, as a consequence of the reduced thickness of the disk of small bodies, $t_{\rm iso} \propto s$ provided that $\alpha^{1/2}s_b < s < s_b$. There is no further reduction of $t_{\rm iso}$ for $s < \alpha^{1/2}s_b$. We note that $t_{\rm iso}$ can be remarkably small. For $s = \alpha^{1/2}s_b$,

$$t_{\rm iso} \sim \begin{cases} 10^4 \text{ yr}, & a = 1 \text{ AU}, \\ 10^5 \text{ yr}, & a = 25 \text{ AU}. \end{cases} \quad (15)$$

This follows from multiplying the values in equation (11) by $\alpha^{1/2}$, since the maximum focusing factor is $\alpha^{-3/2}$ (Greenberg et al. 1991; Greenzweig & Lissauer 1992; Dones & Tremaine 1993).

In the dispersion-dominated regime, $s > s_b$, $u/v_{\rm H} \sim (s/s_b)^{2/9}$. Thus, $u_{\rm col} \sim u$, and the width of an oligarch's feeding zone $u/\Omega \sim (s/s_b)^{2/9}R_{\rm H}$. Consequently, to obtain values for $R_{\rm iso}$, $M_{\rm iso}$, $N_{\rm iso}$, and $t_{\rm iso}$ appropriate to the dispersion-dominated regime, we must multiply those given in equations (8), (9), (10), and (11) by factors of $(s/s_b)^{1/9}$, $(s/s_b)^{1/3}$, $(s/s_b)^{-1/3}$, and $(s/s_b)^{5/9}$, respectively.

2.5. *Summary*

The material in this section is a synthesis of two important ideas: the potential for rapid growth of protoplanets by the accretion of small bodies (e.g., Greenberg et al. 1984; Bryden et al. 2000), and the proposition that with a modest enhancement of surface density above that of the minimum solar nebula, the isolation mass in the outer planet system would have been comparable to the masses of Uranus and Neptune (e.g., Lissauer et al. 1996).

There are several messages to take away from this section:

1. A short isolation timescale requires accretion of small bodies.
2. As the result of viscous stirring by oligarchs, small bodies suffer collisions at velocities $\geq v_{\rm H}$ that we assume are sufficient to fragment them.
3. We note that $t_{\rm iso}$ as used by us measures the duration of the final doubling of an oligarch's mass. It may be much smaller than the duration of orderly or runaway growth, or even the initial phase of oligarchy. Thus, $t_{\rm iso}$ may not mark the age of the solar system at isolation.
4. Models for Uranus and Neptune imply that each planet contains a few Earth masses of hydrogen and helium (Guillot 1999). This is consistent with a formation timescale $\sim 10^7$ yr, which requires that these planets grew by accreting mainly meter-size or smaller bodies.

5. If collisional fragmentation continues to small enough sizes, the disk of small bodies would be optically thick. Then it would be described by fluid rather than by particle dynamics.

6. The isolation mass for the minimum mass solar nebula is smaller than the planet mass, and by a much greater margin in the inner planet system than in the outer.

3. BEYOND OLIGARCHY, $\Sigma > \sigma$

We show below that oligarchy ends when $\Sigma \approx \sigma$. This result applies to accretion in both shear- and dispersion-dominated regimes.

3.1. *Instability of Protoplanet's Velocity Dispersion*

As soon as $\Sigma > \sigma$, the velocity dispersion of the big bodies destabilizes. This occurs because the typical relative velocity between a big and small body is $v > u$, so equations (2) and (5) are modified to[5]

$$\frac{1}{v}\frac{dv}{dt} = \frac{(\Sigma - \sigma)\Omega}{\rho R}\alpha^{-2}\left(\frac{v_{\rm H}}{v}\right)^4. \quad (16)$$

Thus, when $\Sigma > \sigma$, big bodies are heated faster than they are cooled. This marks the end of oligarchy. As v increases, heating and cooling both slow down, but heating always dominates cooling. Eventually the orbits of neighboring big bodies cross.

Because it is based on approximate rates for viscous stirring and dynamical friction, the criterion, $\Sigma \sim \sigma$, for the onset of velocity instability is also approximate. Our choice of 6 times the minimum mass solar nebula surface density in the outer planet region is based on the assumption that $\Sigma = \sigma$ at isolation. If instead, at the onset of velocity instability the oligarchs contained most of the mass, the appropriate enhancement factor would be slightly above 3. *N*-body simulations of oligarch dynamics with the addition of accurate analytic expressions for dynamical friction can resolve this issue.

The consequence of the instability in the velocity dispersion differs according to which is larger, the escape velocity from the surfaces of the planets that ultimately form or the escape velocity from their orbits. The ratio of these two escape velocities is

$$\mathcal{R} \sim \begin{cases} 0.3 & \text{for } a = 1 \text{ AU},\ M_p = M_\oplus, \\ 2.3 & \text{for } a = 25 \text{ AU},\ M_p = 15\ M_\oplus. \end{cases} \quad (17)$$

Before we proceed to discuss these two cases, we stress an essential point that is central to the outcome of each. *N*-body planet systems can possess long-term stability. This behavior lies outside the realm that naive calculations of planetary interactions can describe. We propose that in both cases, $\mathcal{R} < 1$ and $\mathcal{R} > 1$, the system of big bodies evolves such that the surviving planets have close to the smallest spacings allowed by long-term stability.

3.1.1. *Inner Planet System, $\mathcal{R} \ll 1$: Coalescence*

In regions where $\mathcal{R} \ll 1$, the big bodies' velocity dispersion increases until it becomes of the order of the escape velocity from their surfaces. At this point they begin to collide and coalesce. Coalescence slows as the number of big bodies decreases and their individual masses increase.

The timescale for the formation of planet-size bodies with radius R_p whose orbits are separated by of order a is just

$$t_{\rm coag} \sim \left(\frac{\rho R_p}{\sigma\Omega}\right) \sim 10^8 \text{ yr at } a = 1 \text{ AU for } R_p = R_\oplus. \quad (18)$$

At a separation of order a, mutual interactions no longer produce chaotic perturbations. Indeed, detailed *N*-body simulations of terrestrial planet formation by Chambers (2001) produce stable systems on a timescale similar to $t_{\rm coag}$.

What happens to the small bodies while the big ones are colliding and coalescing? A significant fraction of them collide with and are accreted by big bodies. Additional small bodies are created in grazing collisions between big ones (Leinhardt & Richardson 2002). This ensures that a significant residual population of small bodies persists until the end of coalescence.

3.1.2. *Outer Planet System, $\mathcal{R} \gg 1$: Ejection*

In regions where $\mathcal{R} \gg 1$, v reaches the orbital speed Ωa. Some fraction of the big bodies become detached from the planetary system and either take up residence in the Oort Cloud or escape from the Sun. This continues until mutual interactions among the surviving big bodies are no longer capable of driving large-scale chaos.

We estimate the ejection timescale as

$$t_{\rm eject} \sim \frac{0.1}{\Omega}\left(\frac{M_\odot}{M_p}\right)^2 \sim 10^9 \text{ yr at } a = 25 \text{ AU}. \quad (19)$$

Shoemaker & Wolfe (1984) and L. Dones et al. (2004, in preparation) report similar timescales for the ejection of test particles placed on orbits between Uranus and Neptune, the former from a crude impulsive treatment of scattering and the latter from *N*-body integrations. A shorter timescale might apply if bodies were transferred to and then ejected by Jupiter and Saturn. A quantitative estimate of the transfer rate may be obtained from equation (30).

As the random velocity of a big body increases, the rate at which it accretes small bodies declines. Thus, a substantial surface density of small bodies is likely to remain after most of the big bodies have been ejected. In the following section we argue that most of the mass in these small bodies eventually is either injected into the Oort Cloud or escapes from the Sun.

4. COMPLETION

Here we consider processes that took place at sufficiently late times, later than 10^8 yr in the inner planet region and 10^9 yr in the outer, that it seems safe to ignore effects of gas drag.

4.1. *Gap Clearing*

Gaps were not important prior to isolation because the radial spacing of big bodies was only a few times larger than the widths of their feeding zones. But after the protoplanets achieved large-scale orbital stability, their radial spacing was much larger than their Hill radii and wide gaps would have formed around their orbits.

Gap formation is driven by the torque per unit mass (Goldreich & Tremaine 1980)

$$T_p \sim \text{sgn}(x)\left(\frac{M_p}{M_\odot}\right)^2 \frac{\Omega^2 a^6}{x^4} \quad (20)$$

[5] This applies provided that $v < v_{\rm esc}$.

that a protoplanet exerts on material at distance $x = a - a_p$ from its orbit.[6] The gap width increases with time according to

$$\frac{|x|}{a} \sim \left(\frac{M_p}{M_\odot}\right)^{2/5} (\Omega t)^{1/5} \sim \begin{cases} 0.6 t_{\rm Gyr}^{1/5}, & a = 1 \text{ AU}, \\ 0.6 t_{\rm Gyr}^{1/5}, & a = 25 \text{ AU}, \end{cases} \tag{21}$$

where $t_{\rm Gyr} = t/10^9$ yr and we neglect the presence of other planets.

Because the disk's viscosity arises from random motions excited by the protoplanet, the width of the gap is independent of the collision rate. Gap edges are sharp or diffuse depending on whether collisions damp the amplitudes of epicyclic oscillations excited at conjunctions before or after their phases decohere (Borderies et al. 1989). The former would allow accretion, albeit inhibited, while the latter would shut it off altogether. The Appendix provides additional details about the excitation of random motions and the profiles of gap edges.

4.2. *Orbit Regularization*

Either coagulation or ejection is likely to end with the surviving big bodies moving on orbits with eccentricities and inclinations of the order of $\mathcal{R} \sim 0.3$ in the inner planet system and of order unity in the outer planet system. The former is seen in *N*-body simulations of the formation of terrestrial planets from a few hundred big bodies, with no small bodies present (Chambers 2001). Such orbits do not resemble those of solar system planets. In reality, dynamical friction by the residual small bodies tends to circularize and flatten the orbits of the surviving protoplanets. We can compare the rate at which dynamical friction reduces v to that at which big bodies grow by accreting small ones. For $v \gtrsim v_{\rm esc}$ both rates are based on physical collisions and are of the same order. However, for $u < v < v_{\rm esc}$, the rate at which v damps exceeds that at which R grows by the factor $(v_{\rm esc}/v)^2$ for $v_{\rm H} < v < v_{\rm esc}$, $\alpha^{-1}(v/v_{\rm H})$ for $\alpha^{1/2} v_{\rm H} < v < v_{\rm H}$, and $\alpha^{-1/2}$ for $v < \alpha^{1/2} v_{\rm H}$. These comparisons apply to a planet that either cannot or has yet to open a gap around its orbit.

Dynamical friction continues to act after gap opening. Angular momentum and energy are transferred between the planet and the disk of small particles by torques that the planet exerts at Lindblad and corotation resonances. Ward & Hahn (1998, 2003) used the standard torque formula (Goldreich & Tremaine 1980) and concluded that the most potent contributions to the damping of eccentricity and inclination are due to torques at apsidal and nodal resonances. They assessed these contributions to be larger, by factors of $\sim\Omega/|\dot{\varpi}|$ and $\sim\Omega/|\dot{\Omega}_{\rm np}|$, than those from torques at standard first-order corotation and Lindblad resonances.[7] However, this result comes with a number of caveats, especially in applications to disks in which self-gravity dominates pressure in the dispersion relation for apsidal and nodal waves (Goldreich & Sari 2003). Ward & Hahn (1998, 2003) assume that these waves are excited at apsidal and nodal resonances, then propagate away and ultimately damp. They further assume that the resonances lie farther from the planet than the first wavelengths of the waves. However, the main excitation of these waves may occur off resonance at gap edges, and their long wavelengths suggest that they may have more of a standing than a propagating wave character. Each of these features, and especially the latter, is likely to reduce the rates of eccentricity and inclination damping, but by amounts that are difficult to reliably estimate. Our investigation, although inconclusive, suggests that damping of eccentricity and inclination is more likely to occur in particle disks than in gas disks.

4.3. *Cleanup*

What was the fate of the residual small bodies that remained after the protoplanets had settled onto stable orbits? At the end of oligarchy, small bodies and protoplanets contributed comparably to the overall surface density, but today the mass in small bodies is much less than that in planets. The asteroid belt contains most of the mass not in planets inside the orbit of Jupiter, but it totals $\lesssim 10^{-3}\ M_\oplus$. Our knowledge of small bodies in the outer planet region is less complete, but observations of perihelion passages of Halley's comet limit the mass of a disk at $a \gg 30$ AU to be $\lesssim 10(a/100\ {\rm AU})^3\ M_\oplus$ (Hamid et al. 1968; Yeomans 1986; Hogg et al. 1991).

Cleanup was both the last and longest stage in solar system evolution. It is ongoing in both the asteroid belt and Kuiper Belt. The Oort comet cloud was probably populated during this stage. We outline our thoughts on cleanup below. They are speculations based on interweaving theory and observation.

4.3.1. *Direct Accretion of Small Bodies*

Accretion of small bodies by protoplanets is the most obvious mechanism for cleanup. The rate at which a protoplanet gains mass by accreting small bodies with $u \sim v_{\rm H}$ from gap edges at $|x| \lesssim 2.5 R_{\rm H}$ is

$$\frac{1}{M_p}\frac{dM_p}{dt} \sim \frac{\sigma_0}{\rho R_p}\alpha^{-1}\left(\frac{5R_{\rm H}}{\Delta a}\right)^4 \Omega, \tag{22}$$

where Δa is the distance between neighboring planets, σ_0 is the surface density of the small bodies far from the gap, and we assume that the gap's surface density profile obeys $\sigma \propto x^4$ (see eq. [A18]). A more relevant expression is that for $t_{\rm clean} \equiv \sigma_0 |d\sigma_0/dt|^{-1} \approx 2\pi\sigma_0 a \Delta a (dM_p/dt)^{-1}$:

$$\begin{aligned} t_{\rm clean} &\sim 2\times10^{-2}\alpha^{-1}\left(\frac{M_\odot}{M_p}\right)^2\left(\frac{\Delta a}{a}\right)^5 \Omega^{-1} \\ &\sim \left(\frac{\Delta a}{a}\right)^5 \times \begin{cases} 8\times10^{10}\ \text{yr}, & a = 1 \text{ AU}, \\ 7\times10^{11}\ \text{yr}, & a = 25 \text{ AU}. \end{cases} \end{aligned} \tag{23}$$

In both the inner and outer planet system, the spacing between planets is $\Delta a \sim a/3$, so $t_{\rm clean} \sim 300$ Myr for 1 AU and $t_{\rm clean} \sim 3$ Gyr for 25 AU. The latter time is uncomfortably long. It would be a factor $\alpha^{1/2} \sim 1/60$ smaller for $u \lesssim \alpha^{1/2} v_{\rm H}$. However, this introduces a new problem. Maintaining such a low velocity dispersion requires frequent collisions and therefore substantial optical depth. This may lead to sharp gap edges and consequently the absence of accretion. See the Appendix for more discussion of gap structure.

4.3.2. *Second-Generation Planetesimal Formation*

Toward the end of oligarchy, small bodies attain random speeds of the order of 10^2 m s^{-1} (eq. [13]). Collisions at such high speeds fragment them to sizes much smaller than a kilometer. After orbit regularization the protoplanets are spaced by many times their Hill radii and viscous stirring of

[6] Eq. (20) is obtained by a radial smoothing of the torque, which has peaks at mean motion resonances.

[7] The symbols $\dot{\varpi}$ and $\dot{\Omega}_{\rm np}$ denote apsidal and nodal precession rates, respectively.

the intervening small bodies is considerably weaker. An estimate for the rms random velocity of the small bodies is given in equation (A4). With our standard parameters it yields

$$u_{\rm rms} \sim \left(\frac{\rho s}{\sigma}\right)^{1/2} \left(\frac{a}{|x|}\right)^{3/2} \times \begin{cases} 0.1 \text{ m s}^{-1}, & a = 1 \text{ AU}, \\ 0.3 \text{ m s}^{-1}, & a = 25 \text{ AU}, \end{cases} \qquad (24)$$

where $|x|$ is radial distance from the protoplanet. Even these small rms velocities are likely to be much larger than the mean random velocities. That is because the protoplanet's torque is concentrated at discrete mean motion resonances, and the nonlinear disturbances it raises damp locally (Goldreich & Tremaine 1978). These strongly stirred regions near resonances make the dominant contributions to $u_{\rm rms}$. A semiquantitative discussion of this point is provided in the Appendix.

Do larger bodies, referred to here as planetesimals, form under the conditions that prevail after orbit regularization? We are unable to answer this question with confidence. Instead, we critique the difficulties faced by coagulation and gravitational instability, the leading candidates for planetesimal formation.

4.3.2.1. Coagulation

Without gravitational focusing, coagulation is a lengthy process. To double its mass, a body would have to pass through the disk a minimum of $\rho s/\sigma$ times. A potential problem is that a small body's rms random velocity estimated from equation (24) is greatly in excess of the escape velocity from its surface,

$$v_{\rm esc} \sim \alpha^{-3/2}\Omega s \sim \begin{cases} 0.7\left(\frac{s}{\rm km}\right) \text{ m s}^{-1}, & a = 1 \text{ AU}, \\ 0.4\left(\frac{s}{\rm km}\right) \text{ m s}^{-1}, & a = 25 \text{ AU}. \end{cases} \qquad (25)$$

This would imply that collisions lead to disruption rather than to coalescence. Only bodies larger than

$$s_{\rm crit} \sim \frac{M_p^2}{M_\odot M_d}\frac{a^4}{|x|^3} \sim \begin{cases} 5\times10^2 \text{ km}, & a = 1 \text{ AU}, \\ 2\times10^5 \text{ km}, & a = 25 \text{ AU} \end{cases} \qquad (26)$$

have $v_{\rm esc} > u_{\rm rms}$, where, in the numerical evaluation, we have set the disk mass, $M_d \sim \sigma a^2$, equal to the planet mass, M_p, and $|x| = a$.

It might be argued that equation (24) does not apply, that chaotic stirring would not occur far from a planet. This is certainly true for perturbations from a single planet moving on a nearly circular orbit. However, N-body calculations by several groups show that stable orbits between planets are rare; even those initialized with low eccentricities and inclinations invariably become planet orbit crossers (Gladman & Duncan 1990; Holman & Wisdom 1993; Grazier et al. 1999). Nevertheless, there are a couple of reasons to wonder whether coagulation might still occur. None of the N-body calculations investigated the stability of orbits with initial random velocities as small as a few meters per second, and none of them included the small amount of damping that passage through the particle disk would cause.

4.3.2.2. Gravitational Instability

Gravitational instability is another possible mechanism for the formation of second-generation planetesimals. It has the virtue of being very fast. However, it also faces a problem. The formation of solid bodies by gravitational instability requires the particle disk to be optically thick. Observations of thermal infrared radiation from solar-type stars constrain the frequency of protoplanetary systems with optically thick disks.

Suppose that the random velocity of the small bodies falls below the limit for gravitational instability. That is,

$$u \lesssim u_{\rm stab} \sim \frac{\pi G\sigma}{\Omega} \sim \begin{cases} 10 \text{ cm s}^{-1}, & a = 1 \text{ AU}, \\ 1 \text{ m s}^{-1}, & a = 25 \text{ AU}. \end{cases} \qquad (27)$$

Gravitational instabilities convert potential energy into kinetic energy of random motions. The development of nonlinear overdensities requires this energy to be dissipated at the collapse rate $\sim\Omega$. Otherwise, the random velocity dispersion would be maintained near the margin of stability, that is, $u \sim u_{\rm stab}$ (Gammie 2001). Inelastic collisions are the only option for dissipating energy in a particle disk. For the collision rate to match the collapse rate, the particle disk would have to be optically thick, $\sigma/(\rho s) \gtrsim 1$. An optically thick particle disk might result from a collisional fragmentation cascade.

The maximum size of a solid body that can form by collapse without angular momentum loss in a gravitationally unstable disk is

$$s_* \sim \alpha^{-3/2}\frac{\sigma}{\rho} \sim \begin{cases} 50 \text{ m}, & a = 1 \text{ AU}, \\ 2 \text{ km}, & a = 25 \text{ AU}. \end{cases} \qquad (28)$$

Rapid damping of random velocities suggests that this is the size of first bodies that will form by gravitational instability. Since the escape velocity from their surfaces is $u_{\rm stab}$, mutual interactions could maintain their random velocities at an adequate level to stabilize the disk.

4.3.3. *Inner Planet System*

We assume that most of the mass contained in small bodies at the end of coalescence ended up in planets and that only a small fraction fell into the Sun or was ejected by Jupiter. This assumption should be scrutinized, but that is not done here.

The timescale for cleanup by the accretion of small bodies, as given in equation (23), could be comparable to or, for $u < v_{\rm H}$, even shorter than that for coagulation and orbit regularization. However, this should not be taken to imply that second-order planetesimals did not form during cleanup.

4.3.4. *Outer Planet System*

4.3.4.1. Difficulties with Accretion

Accretion of the small bodies would be the simplest solution to cleanup. Estimates based on equation (23), which assumes $u \approx v_{\rm H}$, suggest that it would take a time comparable to the age of the solar system for Uranus and Neptune to clean up the region between them, which has $\Delta a \sim a/3$, and far longer for Neptune to clean up material from outside its orbit where the gap size would be larger (see eq. [21]). Although for $u \lesssim \alpha^{1/2} v_{\rm H}$ the accretion rate would be a factor $\alpha^{-1/2}$ larger, it would require the disk of small bodies to maintain a substantial optical depth. This might result in sharp gap edges and a negligible accretion rate. Given those uncertainties and our crude estimates, we cannot exclude the possibility of accretion.

A more serious issue for our scenario concerns the amount of material that might have been accreted after isolation. Could Uranus and Neptune have acquired most of their mass during cleanup? Suppose the initial surface density was only twice that of the minimum mass solar nebula and that half remained in the form of small bodies at isolation. Then the isolation mass would have been about 1/10 the mass of the outer planets. After a fraction of the big bodies were ejected, dynamical friction from the small ones would have damped the random velocities of the survivors. These would then have resumed accreting small bodies and, once their masses had grown sufficiently, their velocity dispersion would have again become unstable. This cycle would have repeated until all the small bodies were accreted.[8] The end result would not have been very different from that of our preferred scenario, in which the isolation mass equals the planet mass. However, unless the original surface density exceeded twice that of the minimum mass solar nebula, the repeated ejections would have left too little mass to form planets as large as Uranus and Neptune. In addition, without the formation of a second generation of planetesimals, the connection to comets would be lost (see below).

4.3.4.2. Conditions for Ejection

Ejection is the alternative to accretion. Our story implies that up to $\sim 100\ M_\oplus$ of small bodies was ejected in connection with the formation of Uranus and Neptune. Such a large mass ejection aided by Jupiter and Saturn would have been accompanied by a substantial outward migration of Uranus and Neptune (Fernandez & Ip 1984). It might even have moved them outside the orbits of most of the material from which they formed (Levison & Morbidelli 2003).

To examine the conditions needed for ejection, we consider the fate of a small body with radius s, embedded in a sea of bodies with radii $\leq s$, and with total surface density σ. It collides with a total mass of the order of its own on a timescale

$$t_{\rm col} \sim \frac{\rho s}{\Omega \sigma} \sim 2\left(\frac{s}{1\ {\rm km}}\right)\ {\rm Myr\ for\ 25\ AU}. \qquad (29)$$

By comparison, the timescale for a collisionless test particle placed on a low-eccentricity orbit midway between Uranus's and Neptune's orbit to become an orbit crosser is

$$t_{\rm cross} \sim \left(\frac{M_\odot}{M_p}\right)^2 \left(\frac{\Delta a}{2a}\right)^5 \Omega^{-1} \sim 5\ {\rm Myr}. \qquad (30)$$

The above Öpik-type estimate (Öpik 1976) agrees quite well with results from N-body simulations (Gladman & Duncan 1990; Holman & Wisdom 1993; Grazier et al. 1999). Hence, only bodies with s larger than

$$s_{\rm cross} \sim \frac{\sigma}{\rho} \left(\frac{M_\odot}{M_p}\right)^2 \left(\frac{\Delta a}{2a}\right)^5 \sim 2\ {\rm km} \qquad (31)$$

could have become orbit crossers. However, ejection takes much longer than orbit crossing: $t_{\rm eject} \sim 1$ Gyr (eq. [19]). Only bodies larger than

$$s_{\rm eject} \sim 0.1 \frac{\sigma}{\rho} \left(\frac{M_\odot}{M_p}\right)^2 \sim 500\ {\rm km} \qquad (32)$$

could have been ejected by Uranus and Neptune in the presence of a disk of smaller bodies. However, for Jupiter, equations (19) and (32) yield $t_{\rm eject} \sim 10^5$ yr and $s_{\rm eject} \sim 6$ km.

The Oort Cloud is a repository for kilometer-size bodies that probably formed in and were ejected from the outer planet region. Current estimates of the cloud's mass lie in the range 1–10 $M_\oplus$ (Weissman 1996), with the size of pristine comets, which are of the order of a few kilometers, being a major part of the uncertainty. Detailed numerical calculations that follow the ejection of test particles from the outer planet region show that a few percent end up in the Oort Cloud (L. Dones et al. 2004 in preparation). These, together with the observed flux of new comets, are taken to imply that the outer planets ejected a few hundred Earth masses of kilometer-size bodies. Some fraction may have originated in the vicinity of Uranus and Neptune and been transferred via Saturn to Jupiter, which then ejected them.

Simplified treatments by Shoemaker & Wolfe (1984) and Fernandez (1997), as well as N-body simulations by L. Dones et al. (2004, in preparation), show that 50%–80% of test particles initially placed between Uranus and Neptune are, in fact, ejected by Jupiter. However, these investigations did not include collisional damping, whose importance was first recognized by Stern & Weissman (2001) and further investigated by Charnoz & Morbidelli (2003). When this is accounted for, we arrive at stronger result: Jupiter and, to a much lesser extent, Saturn were responsible for ejecting almost all of the kilometer-size bodies into the Oort Cloud. For this scenario to work, kilometer-size bodies must have formed out of the much smaller collisional debris that existed at the end of oligarchy (see § 4.3.2).

5. DISCUSSION

5.1. *Conclusions*

The scenario sketched in this letter addresses some of the basic problems in planet formation:

1. The number and orbital spacing of the planets resulted from an evolution toward stability against large-scale chaotic perturbations.
2. After the cessation of chaotic perturbations, dynamical friction by the residual small bodies damped the orbital eccentricities and inclinations of the surviving protoplanets.
3. Accretion during oligarchy involved small bodies created by a collisional fragmentation cascade. This stage probably lasted for less than 10^5 yr in the inner planet system and less than 10^7 yr in the outer planet system.
4. The timescale for establishing the final configuration of planetary orbits was a few hundred million years for the inner planet system and a few billion years in the outer planet system. It was set by the accretion rate at the geometrical cross section in the former and by the ejection rate at the gravitationally enhanced cross section in the latter.[9]
5. Cleanup of small bodies is a complicated and poorly explored stage of planet formation. Small bodies in the inner planet system were incorporated into planets. Those in the outer planet system were probably ejected by Jupiter and Saturn, but that requires a second generation of planetesimal formation.

5.2. *Influence of Gas*

We have neglected the influence of gas. Observations of young stars indicate that protostellar disks dissipate in a few million years (Haisch et al. 2001; Strom et al. 1993). We show below that although the presence of gas would alter some of our numerical results, it would not affect our picture qualitatively.

Gas drag can provide significant damping for the random velocities of small bodies in addition to that due to inelastic collisions. Relative to collisions, it is most effective in damping

[8] For the sake of argument, we assume that the accretion rate would have been fast enough for this to happen.

[9] The timescale in the outer planet system could have been much shorter if all ejections were done by Jupiter and Saturn.

the random velocities of bodies that are smaller than the mean free path of the gas molecules. For these, its damping rate obeys the same expression as that due to collisions, but with the surface density of the small bodies replaced by that of the gas. We can account for the effects of gas drag by considering s to be an effective size for the small bodies that can be less than their true size. This is a minor point for our story since, as we have emphasized, the true size of the small bodies is highly uncertain. Moreover, our main concern is with the stages of planet formation that follow velocity instability, and these probably continue after the gas is gone.

Rafikov (2003) explored the fast accretion of protoplanetary cores in the presence of gas. His investigation runs parallel to the early phases of ours. However, it terminates at the onset of velocity instability, when $v \sim v_{\rm H}$.

A potentially more significant effect of gas drag that was not considered by Rafikov (2003) is its role in damping the random velocities of the oligarchs. Ward (1993) shows that the gas damping rate can be obtained from the damping rate due to small bodies by substituting the surface density of gas for that of the small bodies and the sound speed of the gas, c_s, for the random velocity of the small bodies, u.[10] By stabilizing the oligarchs' random velocities, gas drag could have enabled them to consume all of the small bodies.

In the inner planet system, it is possible, although highly uncertain, that much of the gas survived until isolation. Then the full velocity instability of the oligarchs would have been delayed until the surface density of gas declined to match that contributed by oligarchs. After that the oligarchs would have excited their random velocities up to their escape speeds. Although most of the small bodies would have been accreted before this happened, plenty of new ones created in glancing collisions could have damped the orbital eccentricities and inclinations of the planets that finally formed.[11]

Outer solar system planets, Uranus and Neptune, are believed to have collected only a few Earth masses of nebular gas. Thus, it is likely that most of the gas had disappeared prior to isolation in the outer planet system.

Gas drag must have been more significant in the formation of Jupiter and Saturn. One might worry that the orbital decay of small particles, which are an integral part of our scenario, would have been too fast for them to have been accreted. Particles with stopping time comparable to their orbital time drift fastest. Their orbits decay on a timescale $\Omega^{-1}(a\Omega/c_s)^2 \sim 10^3$ yr. By damping the random velocities of small bodies, gas drag can protect them from undergoing destructive collisions. This may result in larger bodies, for which the drift timescales are longer. Moreover, gas drag could have made the isolation timescale in the Jupiter-Saturn region as short as $\sim 10^4$ yr (see eq. [15]). Another effect of gas would be to enhance the accretion cross section of a massive protoplanet by forming a dense envelope around it (Inaba et al. 2003).

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[10] We note that at isolation $v_{\rm H} < c_s < v_{\rm esc}$. Moreover, we are assuming that $v < c_s$.

[11] Since it took the inner planets more than 100 Myr to form (eq. [18]), gas is unlikely to have contributed to regularizing their orbits.

APPENDIX

In this appendix we consider how a planet affects a disk of small bodies in which it is embedded. We assume throughout that the collision time between small bodies is longer than the time it takes their epicyclic phases to decohere. This applies provided that the optical depth of the small bodies at a distance $|x| > R_{\rm H}$ from the planet is less than $\tau_{\rm crit} \sim (R_{\rm H}/|x|)^3$. If this condition does not hold, i.e., if $\tau > \tau_{\rm crit}$, then the effective viscosity would be negative, and the planet would open a gap with sharp edges (Borderies et al. 1989).

A1. STIRRING

To determine the mean random kinetic energy of the small bodies, we balance their energy loss rate in inelastic collisions, $(1-\epsilon^2)u^2/t_{\rm col}$, against the sum of the rates of energy gain from direct forcing by the planet, $(\Omega_p - \Omega)T_p$ (Borderies et al. 1982), plus viscous dissipation acting on the Keplerian shear, $\nu(a d\Omega/da)^2$. The symbol $\epsilon < 1$ denotes the coefficient of restitution, assumed to be a decreasing function of impact velocity, Ω_p is the orbital frequency of the planet, and T_p is the torque per unit mass exerted by the planet. For $|x| \ll a$ we obtain

$$\left(1-\epsilon^2\right)\frac{u^2}{t_{\rm col}} \approx \frac{3x}{2a}\Omega T_p + \frac{9}{4}\nu\Omega^2, \tag{A1}$$

where the kinematic viscosity

$$\nu \sim \frac{u^2}{t_{\rm col}}\Omega^{-2} \tag{A2}$$

(Goldreich & Tremaine 1978),[12] and

$$t_{\rm col}^{-1} \sim \frac{\sigma\Omega}{\rho s} \approx \tau\Omega. \tag{A3}$$

[12] We consider only the case $\Omega t_{\rm col} > 1$; more generally, Goldreich & Tremaine (1978) show $\nu \sim u^2 t_{\rm col}[1 + (\Omega t_{\rm col})^2]^{-1}$ for circular, Keplerian rotation.

In the limit $T_p = 0$, energy released from the Keplerian shear would maintain u at some small value set by the velocity dependence of ϵ. The velocity dispersion in unperturbed parts of Saturn's rings, $u \lesssim 1$ cm s^{-1}, is a practical example. Our interest is in circumstances under which forcing by a planet results in an equilibrium value of u that is much larger than that produced solely by viscous dissipation acting on the Keplerian shear. Under such conditions

$$u \sim \frac{M_p}{M_\odot}\left(\frac{\rho s}{\sigma}\right)^{1/2}\left(\frac{a}{|x|}\right)^{3/2}\Omega a. \tag{A4}$$

To this point we have proceeded as though all small bodies were of the same size. However, equation (A4) applies more generally and yields the size dependence of the rms random velocity of a small body subject to two limits. It must be larger than those that contain most of the mass but small enough so that gravitational focusing does not enhance its interactions with them. In this generalized interpretation, σ must be interpreted as the total surface density in small bodies of all sizes.

A2. RESONANCES

The planet's torque is concentrated at resonances. Equation (A4) is a spatial average of the torques at mean motion resonances. These dominate the heating of the random velocities of small bodies. As the rms random velocity given by equation (A4) is a spatial average, it is of limited utility.

A more complete picture is obtained by investigating the disturbances raised by torques at individual principal mean motion resonances (Goldreich & Tremaine 1980). Our starting point is the WKB dispersion relation for non-axisymmetric waves of angular degree m in a cold, self-gravitating disk:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\sigma|k|. \tag{A5}$$

For principal mean motion resonances, $\omega = m\Omega_p$, where the pattern speed, Ω_p, is equal to the planet's mean orbital angular velocity. At the Lindblad resonance, $|k| = 0$ and $|x|/a \approx 2/(3m)$.

Density waves are excited at each mean motion resonance. Their properties have been extensively studied; we merely quote a few relevant results (Goldreich & Tremaine 1978). These are specialized to the case of a near Keplerian disk for which $\kappa \approx \Omega$. A wave propagates away from the resonance and the planet at the group speed

$$v_g = \frac{\pi G\sigma}{\Omega}. \tag{A6}$$

Its first wavelength

$$\frac{\lambda_1}{|x|} = \left(\frac{\sigma a^2}{M_\odot}\frac{a}{|x|}\right)^{1/2}. \tag{A7}$$

At each encounter with the protoplanet, a disk particle receives a kick sufficient to change its orbital eccentricity by

$$\Delta e \sim \frac{M_p}{M_\odot}(a/|x|)^2. \tag{A8}$$

At a Lindblad resonance, successive increments in e sum coherently over a time comparable to that during which a disturbance propagating at the group velocity crosses the first wavelength. The number of encounters that occur in this time is

$$N \sim \frac{\lambda_1\Omega}{v_g}\frac{|x|}{a}, \tag{A9}$$

so that at resonance

$$e_{\rm res} \sim N\Delta e \sim \left(\frac{M_p^2}{M_\odot\sigma a^2}\frac{a}{|x|}\right)^{1/2}. \tag{A10}$$

The nonlinearity of the wave, $\Delta\sigma/\sigma$, is of the order of the ratio of the coherent epicyclic excursions to the wavelength. Near resonance this gives

$$\frac{a}{\lambda_1}e_{\rm res} \sim \frac{M_p}{\sigma a^2}\frac{a}{|x|}. \tag{A11}$$

In our scenario, the disk and planet have comparable masses, and both are much smaller than the solar mass. Thus, the density waves reach order unity nonlinearity within their first wavelengths and their first wavelengths are much smaller than the distance between neighboring resonances. Therefore, the waves only propagate a small fraction of the distance between resonances before damping. All of the planet's excitation of random velocities is concentrated in these narrow regions where they damp.

A3. GAPS

A protoplanet clears a gap in the disk of small bodies in which it is embedded. Epicyclic motions are excited when small bodies pass conjunction with the protoplanet. Provided that their phases decohere before collisions damp their amplitudes, the gap edges will be diffuse rather than sharp. Since the accretion rate in the shear-dominated limit is proportional to the surface density at $|x| \approx 2.5R_{\rm H}$, it is important to determine the gap's surface density profile. In order to do so, we must estimate the order unity coefficient relating the kinematic viscosity to the rate at which energy per unit mass is dissipated by inelastic collisions. We define b through

$$\nu\Omega^2 = \frac{4b}{9}\frac{u^2}{t_{\rm col}}. \tag{A12}$$

Combining equations (A1) and (A12), we find

$$b = 1 - \epsilon_*^2, \tag{A13}$$

where ϵ_* is the value of ϵ at which the equilibrium velocity dispersion is obtained for stirring by the Keplerian shear in the absence of a protoplanet. Since $T_p = K/x^4$ (eq. [20]), equation (A1) also implies that

$$\frac{u^2}{t_{\rm col}} \approx \frac{3\Omega K}{2a\left(\epsilon_*^2 - \epsilon^2\right)|x|^3}. \tag{A14}$$

In steady state, the torque per unit mass that the protoplanet exerts on a small body, K/x^4, is balanced by the viscous torque. Thus,

$$\frac{K}{x^4} = \frac{3\Omega a}{2\sigma}\frac{d(\nu\sigma)}{dx}, \tag{A15}$$

where we neglect gradients of Ω and a, since they are much smaller than those of $\nu\sigma$ for $|x| \ll a$. Combining equations (A12), (A13), (A14), and (A15) gives

$$\frac{d}{dx}\left[\frac{\left(1-\epsilon_*^2\right)}{\left(\epsilon_*^2-\epsilon^2\right)}\frac{\sigma}{|x|^3}\right] = \frac{\sigma}{x^4}. \tag{A16}$$

For circumstances where stirring by the protoplanet is much greater than that due to the Keplerian shear, it is likely that $\epsilon \ll \epsilon_*$. In this case equation (A16) yields the gap profile

$$\sigma \propto |x|^q, \qquad \text{with } q = 3 + \frac{\epsilon_*^2}{\left(1-\epsilon_*^2\right)}. \tag{A17}$$

It is difficult to obtain a reliable estimate for ϵ_*. Goldreich & Tremaine (1978) obtained an approximate solution of the collisional Boltzmann equation for a model in which the particles were represented by smooth spheres separated by many times their diameters. They found $\epsilon_* \approx 0.63$, which implies

$$\sigma \propto |x|^{3.66}. \tag{A18}$$

For the purposes of the present paper, we are content to approximate this as $\sigma \propto |x|^4$.

It is interesting to compare the above gap profile with earlier results. Lissauer et al. (1981) assumed a constant velocity dispersion, which leads to a clean gap and a much steeper edge profile. Borderies et al. (1982, 1989) showed that a clean gap is inconsistent with enhanced stirring by a planet unless streamline distortion giving rise to a local reversal of the direction of viscous transport of angular momentum is taken into account.

REFERENCES

Borderies, N., Goldreich, P., & Tremaine, S. 1982, Nature, 299, 209
———. 1989, Icarus, 80, 344
Bryden, G., Lin, D. N. C., & Ida, S. 2000, ApJ, 544, 481
Chambers, J. E. 2001, Icarus, 152, 205
Charnoz, S., & Morbidelli, A. 2003, Icarus, 166, 141
Dones, L., & Tremaine, S. 1993, Icarus, 103, 67
Fernandez, J. A. 1997, Icarus, 129, 106
Fernandez, J. A., & Ip, W.-H. 1984, Icarus, 58, 109
Gammie, C. F. 2001, ApJ, 553, 174
Gladman, B., & Duncan, M. 1990, AJ, 100, 1680

Goldreich, P., Lithwick, Y., & Sari, R. 2004, ARA&A, 42, 549
Goldreich, P., & Sari, R. 2003, ApJ, 585, 1024
Goldreich, P., & Tremaine, S. D. 1978, Icarus, 34, 227
———. 1980, ApJ, 241, 425
Grazier, K. R., Newman, W. I., Varadi, F., Kaula, W. M., & Hyman, J. M. 1999, Icarus, 140, 353
Greenberg, R., Bottke, W. F., Carusi, A., & Valsecchi, G. B. 1991, Icarus, 94, 98
Greenberg, R., Weidenschilling, S. J., Chapman, C. R., & Davis, D. R. 1984, Icarus, 59, 87
Greenzweig, Y., & Lissauer, J. J. 1992, Icarus, 100, 440
Guillot, T. 1999, Science, 286, 72
Haisch, K. E., Lada, E. A., & Lada, C. J. 2001, ApJ, 553, L153
Hamid, S. E., Marsden, B. G., & Whipple, F. L. 1968, AJ, 73, 727
Hayashi, C. 1981, Prog. Theor. Phys. Suppl., 70, 35
Hogg, D. W., Quinlan, G. D., & Tremaine, S. 1991, AJ, 101, 2274
Holman, M. J., & Wisdom, J. 1993, AJ, 105, 1987
Inaba, S., Wetherill, G. W., & Ikoma, M. 2003, Icarus, 166, 46
Kokubo, E., & Ida, S. 1998, Icarus, 131, 171
Leinhardt, Z. M., & Richardson, D. C. 2002, Icarus, 159, 306
Levison, H. F., & Morbidelli, A. 2003, Nature, 426, 419
Lissauer, J. J. 1987, Icarus, 69, 249
———. 1993, ARA&A, 31, 129
Lissauer, J. J., Pollack, J. B., Wetherill, G. W., & Stevenson, D. J. 1996, in Neptune and Triton, ed. D. P. Cruikshank (Tucson: Univ. Arizona Press), 37
Lissauer, J. J., Shu, F. H., & Cuzzi, J. N. 1981, Nature, 292, 707
Öpik, E. J. 1976, Interplanetary Encounters: Close-Range Gravitational Interactions (Amsterdam: Elsevier)
Petit, J.-M., & Henon, M. 1986, Icarus, 66, 536
Rafikov, R. R. 2003, preprint (astro.ph/0311440)
Safronov, V. S. 1972, Evolution of the Protoplanetary Cloud and Formation of the Earth and Planets (NASA TTF-677; Moscow: Nauka)
Shoemaker, E. M., & Wolfe, R. F. 1984, Lunar Planet. Sci. Conf., 15, 780
Stern, S. A., & Weissman, P. R. 2001, Nature, 409, 589
Strom, S. E., Edwards, S., & Skrutskie, M. F. 1993, in Protostars and Planets III, ed. E. H. Levy & J. I. Lunine (Tucson: Univ. Arizona Press), 837
Ward, W. R. 1993, Icarus, 106, 274
Ward, W. R., & Hahn, J. M. 1998, AJ, 116, 489
———. 2003, AJ, 125, 3389
Weissman, P. R. 1996, in ASP Conf. Ser. 107, Completing the Inventory of the Solar System (San Francisco: ASP), 265
Wetherill, G. W., & Stewart, G. R. 1989, Icarus, 77, 330
Yeomans, D. K. 1986, in Proc. 20th ESLAB Symp. on the Exploration of Halley's Comet, Vol. 2: Dust and Nucleus ed. B. Battrick, E. J. Rolfe, & R. Reinhard (ESA SP-250; Noordwijk: ESA), 419
Youdin, A. N., & Shu, F. H. 2002, ApJ, 580, 494