Ge/Ay133 – Problem Set #6
Revenge of the (Geo)Chemists
Due November 17^{th}

(1) This problem is to help you think about the thermal history of bodies that are assembled in the early solar system. Information of this sort is important when thinking about the core-instability model of Jovian planet formation and also about comets, asteroids and the differentiation of planetesimals and/or oligarchs.

(a) Show that the gravitational potential energy of a spherical body of uniform density is given by

$$-\frac{3GM^2}{5R} \quad .$$

Assuming a constant ρ and C_p, and given no energy loss from the system, use this fact to generate an equation for the temperature of a body in terms of these constants. What does this relationship predict for the temperature for the moon and for the earth due to accretional energy alone? Why is this an upper limit to the temperature? For both, use a specific heat of $C_p = 10^7$ erg/g/K, and use densities of $\rho = 3.3$, 5.5 g cm^{-3} for the Moon and Earth.

(b) In reality, only a fraction of the accretional energy is trapped as heat in the growing body. Let's assume this efficiency of trapping, ϵ, is something like 2.5%. Recalculate the equations from (a), what temperatures do you derive? The largest asteroid, Ceres, has a radius of 487 km. Assuming a uniform density of $\rho = 3$ g cm^{-3}, what temperature would you expect for Ceres shortly after accretion? If silicates begin to melt at a temperature of $\sim$1500 K, would Ceres be differentiated? The Moon? The Earth?

(c) Now for a little (nuclear) chemistry... Chondritic meteorites are the only direct samples we have of relatively unaltered rocks from the early history of the solar system. A couple of lines of evidence suggests that the parent bodies of these meteorites were at least partially molten: igneous textures and the separation of silicates from metallic phases. The accretional energy analysis above suggests that these bodies should not be differentiated, and there is also insufficient energy from the long lived radionuclides that supply much of the internal heat of the present day earth (^{40}K, $^{235/238}$U, and ^{232}Th). In 1955, Harold Urey suggested that the beta decay of the short-lived isotope ^{26}Al could produce sufficient heat. The reaction is:

$$^{26}\mathrm{Al} \rightarrow ^{26}\mathrm{Mg} \quad ,$$

with a half-life of 750,000 years. If a meteorite contained some ^{26}Al initially, what percent of the ^{26}Al remains in the meteorite today, 4.5 AE later?

(d) Below is a table of published data on the isotopic composition of minerals in chondrule from the Allende meteorite:

Phase	Formula	^{27}Al/^{24}Mg	^{26}Mg/^{24}Mg
Anorthite	$CaAl_2Si_2O_8$	245	0.1517
Anorthite	$CaAl_2Si_2O_8$	128	0.1468
Melilite	$Ca_2(Mg,Al,Si)_3O_7$	9.1	0.1404
Spinel	$MgAl_2O_4$	2.5	0.1398
Fassaite	$Ca(Mg,Al,Ti)(Si,Al)_2O_6$	2.0	0.1398

^{27}Al, ^{26}Mg, and ^{24}Mg are stable isotopes. Neither ^{27}Al nor ^{24}Mg are decay products. Plot ^{26}Mg/^{24}Mg versus ^{27}Al/^{24}Mg and draw a least-squares line through the data points. Calculate the initial ^{26}Al/^{27}Al and initial ^{26}Mg/^{24}Mg of the chondrule.

(e) Using this initial isotopic ratio (of ^{26}Al/^{27}Al) of the chondrule, write an expression for the rate of energy release per gram of meteorite due to the ^{26}Al decay as a function of time. The energy released per decay of an ^{26}Al atom is 3.3 MeV. The average abundance of Al in a chondrite is 0.868% by mass and the only isotopes of Al initially present are ^{26}Al and ^{27}Al. Plot your results.

(f) Finally, let's do a slightly more realistic temperature calculation including radioactive heating. Assume that the asteroid is chemically and isotopically homogeneous with a density of 3.7 g/cm^3, and that the outward energy flux as a function of radius is $F = -k(dT/dr)$ (that is, is proportional to the radial temperature gradient), where the thermal conductivity k equals 3.25 J/s·m·K.

First, write an ordinary differential equation that relates the temperature to radius (in one dimension). Assume the outer boundary condition is the equilibrium temperature for a perfect blackbody radiating away to free space. Second, use the initial heat production (energy per mass per unit time) calculated above to estimate the temperature at the *center* of an asteroid as a function of its radius. Plot your result. If the melting point at the center of the asteroid is ∼1500 K, what is the smallest radius at which the asteroid melts at the center?

You should find this to be a much smaller number than in part (b). So, let's think about time. Estimate the temperature versus radius for an asteroid given a 1 million year and a 10 million year interval for ^{26}Al decay (that is, assume a window between the initial injection of live ^{26}Al and the bulk of accretion of 1 MYr as one test case and then a delay of 10 MYr as a second situation). For these cases, what is the minimum radius of an asteroid that melts at the center? The evidence is that planetesimals smaller than 100 km experienced partial melting. What does these idealized models tell you about the timescale for the assembly of such bodies in the solar nebula?