The entropy of mixing term is always negative (since \( \ln(a) < 0 \) if \( a < 1 \)), so as \( T \to 0 \), the entropy of the mixture goes to:

\[
G = N_1 G_1 + N_2 G_2 + \frac{N_1 N_2 \Delta G}{N_1 + N_2}
\]

For \( \Delta G > 0 \), \( G \) is always greater than the total entropy of the unmixed phases (which is \( N_1 G_1 + N_2 G_2 \)). Since the Gibbs energy of the unmixed phases will always be lower than the entropy of the mixed phases (for low \( T \)), the system will exhibit phase separation.

b. We want the temperature at which:

\[
G_{mix} \leq G_{separate}
\]

for all \( N_1, N_2 \).

\[
N_1 G_1 + N_2 G_2 + \frac{N_1 N_2 \Delta G}{N_1 + N_2} + k_B T \left[ N_1 \ln \left( \frac{N_1}{N_1 + N_2} \right) + N_2 \ln \left( \frac{N_2}{N_1 + N_2} \right) \right]
\]

\[
\leq N_1 G_1 + N_2 G_2
\]
$$k_B T_{\text{phase transition}} = \frac{-N_1 N_2 \Delta G}{(N_1 + N_2) \left[ N_1 \ln \left( \frac{N_1}{N_1 + N_2} \right) + N_2 \ln \left( \frac{N_2}{N_1 + N_2} \right) \right]}$$

This defines the phase transition curve for some $N_1$ & $N_2$. We want the temp above which, for any $N_1$ & $N_2$, both phases will mix.

It’s apparent from the phase curve plotted below that the maximum $T_{\text{phase}}$ occurs at $x_i = \frac{N_1}{N_1 + N_2} = 0.5$.

So to get $T_c$, we evaluate the above at $x_i = x_2 = 0.5$.

$$T_c = \frac{-0.5 \times 0.5 \Delta G/k_B}{0.5 \ln (0.5) + 0.5 \ln (0.5)}$$

Here’s the phase diagram:

![Phase diagram](image)
c. The number fraction of Ne atoms is
\[
\frac{1}{6} \text{(cosmic Ne fraction)} = \frac{1}{6} \left(1.1 \times 10^{-4}\right),
\]
and the number fraction of hydrogen is 0.92 (assumed same as cosmic abundance). Plugging these values into \( \text{A} \) \((x_1 = 1.8 \times 10^{-5}/(0.92 + 1.8 \times 10^{-5}) = 1.9 \times 10^{-5},
\]
x_2 = 0.92/(0.92 + 1.8 \times 10^{-5}) = 0.99998\) at the temperature in Jupiter’s atmosphere (6000 K), we can solve for the minimum possible \( \Delta G \) required for the two phases to be insoluble (“separate phases” above):

\[
\Delta G_{\text{min}} = 6.1 \text{ eV}
\]

Keep in mind that this is ~ the strength of covalent bonds in rocks! This is a very high \( \Delta G \) for mixed gases.

8.2.a. Equation 10 in the chapter gives a diffusion equation for conductivity:

\[
\sigma = \frac{n e^2 \zeta}{m}
\]
m is the mass of the charge carrier (protons in this case), e is the proton charge, \( n \) is the number density of charge carriers, and \( \tau \) is the relaxation time.

We're given that the density is 2 g/cm\(^3\), and we have one free proton per water molecule. So:

\[
n = \frac{e}{m_{\text{H}_2\text{O}} \cdot m_p}
\]

\[
= \frac{1}{16} \quad \text{(or 18 including two H, but oxygen dominates)}
\]

\[
= 6.6 \times 10^{22} \text{ cm}^{-3} = 6.6 \times 10^{28} \text{ m}^{-3}
\]

Next, let's get \( \tau \). To get a time, we need a distance and a velocity. The distance is the internuclear spacing, which is \( n^{-1/3} \Rightarrow \)

\[
\ell \equiv n^{-1/3} = 2 \times 10^{-8} \text{ cm} 
\]

(makes sense: 1 Å = 10^{-8} cm

The velocity is the thermal velocity. There's a rigorous way to do this (using a Maxwell distribution), but the OoM way to get the thermal velocity is:

\[
\frac{1}{2} m_p v_{th}^2 \sim k_B T
\]

\[
v_{th} \sim \sqrt{\frac{2k_B T}{m_p}}
\]
Oh! Now let's get the characteristic time:

$$\tau = \frac{\lambda}{v_{th}} = \frac{2 \times 10^{-9} \text{ cm}}{10^6 \text{ cm/s}} = 2 \times 10^{-14} \text{ s}$$

Putting it all together (using units Dave indicates):

$$\sigma \approx \frac{(6.6 \times 10^{29} \text{ m}^{-3}) (1.6 \times 10^{-19} \text{ C})^2 (2 \times 10^{-14} \text{ s})}{(1.6 \times 10^{-27} \text{ kg})}$$

$$= 2 \times 10^9 \Omega^{-1} \text{ m}^{-1}$$

$$= \boxed{200 \, \Omega \cdot \text{cm}^{-1}} \quad \text{(same OoM as figure!)}$$

b. Now we use the mean Fermi energy to calculate the mean velocity of the charge carriers:

$$E_F = \frac{2.21}{r_s^2} \text{ Ryd} \quad \text{[eq 3.20]}$$
Let's get \( r_s \) using the given density value.

\[
a_0 r_s = \left( \frac{\mu m_H}{e} \right)^{1/3} \quad \text{**\( \mu = 2 \) since we're considering \( \text{H}_2 \)}
\]

\[
rs = 2.8
\]

OK, next set \( E_F \) equal to something with velocity in it to get the mean velocity of charge carriers:

\[
E_F = \frac{1}{2} m e V_F^2
\]

\[
\frac{2.21}{r_s^2} \text{ Ryd} = \frac{e}{2} m e V_F^2
\]

\[
\text{**1 Ryd} = 13.6 \text{ eV} = \frac{13.6}{6 \times 10^{-10}} \text{ erg}
\]

\[
\Rightarrow \quad V_F = \left[ \frac{2 \sqrt{2.21}}{r_s} \times \frac{13.6}{6 \times 10^{-10} \times m_e} \text{ erg} \right]^{1/2}
\]

\[
= 10^8 \text{ cm/s}
\]

[Leaves faster than speed above, but still 100x slower than light speed. Makes sense that electrons move faster since they're less massive.]

We have a speed, we have a length (\( r_s a_0 \))... let's get a time!
\[ Z = \frac{1}{V_F} \]
\[ = \frac{2.8a_0}{10^8 \text{ cm/s}} \]
\[ \approx 10^{-16} \text{ s} \]

All right, we have everything we need to calculate conductivity.

\[ \sigma = \frac{n e^2 c}{m_e} \]
\[ = \left( \frac{\ell}{n m_p} \right) e^2 c \]
\[ \approx \frac{Me}{(3 \times 10^{29} \text{ m}^{-3})(1.6 \times 10^{-19} \text{ C})^2 (10^{-16} \text{ s})} \]
\[ \approx 9 \times 10^{-31} \text{ kg} \]

\[ \approx 9 \times 10^5 \Omega^{-1} \text{ m}^{-1} \]
\[ = 9 \times 10^3 \Omega^{-1} \text{ cm}^{-1} \]

Pretty close to results at highest pressure in figure.
8.4.a. \( \frac{GM^2}{R} \sim MC \Delta T \)

\[ \Delta T \sim \frac{GM}{AC} \]

Plug in \( R = 50 \text{ km}, C = 8 \times 10^6 \text{ ergs/g/K} \),
and \( M = 4 \rho R^3 = 1.7 \times 10^{21} \text{ g} \).

\[ \boxed{\Delta T \sim 3 \text{ K}} \]

\( \nu \) small, ok to ignore.

b. For this problem, \( r = 0 \) @ the center and \( r = R \)
at the surface. Initial temp is 0 everywhere
\( (T(r, t=0) = 0) \) and surface temp is always
0 \( (T(r=R, t) = 0) \).

\[ \int_0^t \frac{\partial T(r;o,t)}{\partial t} dt + \int_0^\infty \left[ \frac{\partial}{\partial t} T(r;o,t) + \frac{4}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T(r;o,t)}{\partial r} \right) \right] d+ \]

\[ T(r;o,t) = A \int_0^\infty D^2 T(r;o,t) d+ - \left[ A T_m \exp \left( -\frac{t}{\tau} \right) \right] \]

\[ T(r;o,t) = A T_m \left( 1 - e^{-\frac{t}{\tau}} \right) + A \int_0^t D^2 T(r;o,t) d+ \]

\( = 0 @ \text{ center} \)
\[ T(r=0,t) = AT_m (1 - e^{-t/\tau}) \leq 1 \text{, so } T(r=0,t) \leq AT_m \]

and if \( A < 1 \), then
\[ T(r=0,t) \leq T_m . \]

c. \[ O = KD^2 T + AT_m \frac{e^{-t/\tau}}{z} \]

By symmetry, \( T \) must be independant of \( \theta \) and \( \varphi \), so
\[ D^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) . \]

\[ O = k \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \right] + \frac{AT_m}{z} e^{-t/\tau} \]

Let's try separation of variables.
\[ T(t,r) = C(r) B(t) . \]

Then:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial C}{\partial r} \right) = B \left[ \frac{\partial^2 C}{\partial r^2} + \frac{2}{r} \frac{\partial C}{\partial r} \right] . \]

Plugging this back into our equation gives:
\[ \frac{\partial^2 C}{\partial r^2} + \frac{2}{r} \frac{\partial C}{\partial r} = -AT_m \exp \left( -\frac{t}{\tau} \right) \]
Now assume $C(r) = -r^2 + d$ and plug in to solve for $x$:

$$-2 - 4 + \lambda = 0$$

$$\Rightarrow \lambda = 6$$

The boundary conditions require that $C(r=\lambda)|_{\lambda=0}$, so $d = \lambda^2$. We now have an expression for $C(r)$:

$$C(r) = \lambda^2 - r^2$$

Next we can get $B(t)$ using our equation for $\lambda$:

$$-\lambda = -\frac{ATn}{BCK} \exp(-t/\tau)$$

$$\downarrow$$

$$\lambda = 6$$

$$B = \frac{ATn}{6CK} \exp(-t/\tau)$$

Putting everything together, we have:

$$T(r,t) = C(r)B(t)$$
\[
\frac{A T_m (R^2 - r^2)}{6 \pi \chi} \exp \left(-\frac{r}{\chi}\right)
\]

d. \quad R_c = \sqrt{6 \pi \chi}

= \sqrt{6 (0.01 \text{ cm}^2/\text{s}) (10^6 \text{ yr}^{-1})}

= 14 \text{ km}