3.4. a. (i) If the moon had constant density, the pressure would be (from hydrostatic equilibrium):

\[ \mathrm{d}P = -p \mathrm{d}g \mathrm{d}r = -p \frac{G \mathrm{m}(r)}{r^2} \mathrm{d}r = -4p^2 Gr \mathrm{d}r \]

\[ \int \mathrm{d}P = \int -4p^2 Gr \mathrm{d}r \]

\[ P(r) - P(R) = -4Gp^2 \int_0^r \frac{1}{r^2} \mathrm{d}r = \frac{4Gp^2}{r} \int_0^r \mathrm{d}r \]

\[ P(r) = 2Gp^2 (R^2 - r^2) \]

(ii) The definition of the modulus is:

\[ K = \frac{1}{2} \frac{\mathrm{d}P}{\mathrm{d}P} \]

Use \( P(r) \) from first part

\[ \int \mathrm{d}P = \int 4Gp^2 (R^2 - r^2) \mathrm{d}r \]

Assuming \( K \) is constant, we can solve for \( p(r) \):

\[ p(r) = \left[ \frac{K}{4G(R^2 - r^2)} \right]^{1/2} \]

(iii) We can use this density distribution to compute the moment of inertia as a function of \( K \).

The moment of inertia of a thin shell is:

\[ \mathrm{d}I = \frac{2}{3} \mathrm{d}m r^2 \]
So let's integrate this from 0 to R to get the total moment of inertia:

\[ I = \int \frac{8}{3} \pi \rho(r) r^4 \, dr \]

\[ = \frac{8\pi}{3} \sqrt{\frac{k}{4G}} \int_0^R \frac{r^4 \, dr}{(R^2 - r^2)} \]

\[ = \frac{\pi^2 R^4}{2} \sqrt{\frac{k}{4G}} \]

Equate our expressions for moment of inertia will give us the bulk modulus needed to produce the moon's moment of inertia:

\[ 0.391MR^2 = \frac{\pi R^4}{2} \sqrt{\frac{k}{4G}} \]

\[ \Rightarrow k = \left[ \frac{4 \times 0.391 M \sqrt{G}}{\pi R^2} \right]^2 \]

\[ = G \left[ 0.498 \times 4 \pi \bar{\rho} \right]^2 \]

Plugging in \( \bar{\rho} = 3.34 \text{ g/cm}^3 \) and \( R = 1760 \text{ km} \) gives:

\[ k = 7 \times 10^8 \text{ g}^{-1} \text{ cm}^3 \text{ s}^2 \left( 4 \times 0.498 \times 1760 \text{ km} \times 3.34 \text{ g/cm}^3 \right)^2 \]
\[
= 9.6 \times 10^{10} \text{ g cm}^{-1} \text{ s}^{-2}
\]
\[
= \frac{9.6 \times 10^{10}}{9.8} \text{ dynes cm}^{-2} \approx 10^5 \text{ bar}
\]

Pretty small / unphysical given the actual bulk modulus of moon material revealed in part b!

b. Now, let's instead assume:

\[
p(r) = \begin{cases} p_{Fe}, & 0 \leq r \leq R_e \\ p_r, & R_e \leq r \leq R \end{cases}
\]

and use it to recalculate the moment of inertia as a function of \( R_e \):

\[
I = \int_0^R 8 \pi r^4 p(r) \, dr
\]

\[
= 8 \left[ \int_0^{R_e} p_{Fe} r^4 \, dr + \int_{R_e}^R p_r (R^5 - R_e^5) \, dr \right]
\]

Equating this expression with the observed moment of inertia gives:

\[
0.391 \, MR^2 = \frac{8}{5} \left[ p_{Fe} R_e^5 + p_r (R^5 - R_e^5) \right]
\]

The total mass of the moon given this new model is:
\[ M = \rho_{Fe} \left( 4R_e^3 \right) + \rho_r \left( 4R^3 - 4R_e^3 \right) \]

Plugging this back into the equality above lets us solve for \( R_e \)...

\[ 4 \times 0.391 \rho_{Fe} R_e^3 + 4 \times 0.391 \rho_r R_e^3 - 4 \times 0.391 \rho_r R_e^3 = \frac{8}{5} \rho_{Fe} R_e^3 \]

\[ R_e^3 \left[ 1.564 \rho_e^2 (\rho_{Fe} - \rho_r) - \frac{8}{5} \rho_e^2 (\rho_{Fe} - \rho_r) \right] = \rho_r R_e^5 \left( \frac{8}{5} - 1.564 \right) \]

Plugging in numbers for \( \rho, \rho_{Fe}, \rho_r \) gives:

\[ R_e^3 \left[ 1.8 \text{ g/cm} - (5.9 \text{ g/cm}^3) R_e^2 \right] = 2 \times 10^{40} \text{ cm}^2 \cdot \text{g} \]

\[ \downarrow \text{ wolfram alpha} \]

Wolfram alpha gives three solutions to this equation:

\[ R_e = \begin{cases} \text{(reject } \Rightarrow \text{ must be } > 0) \left\{ \begin{array}{l} -1758 \text{ km} \\ 496 \text{ km} \end{array} \right. \\
1721 \text{ km} \text{ (ding ding ding!)} \end{cases} \]

\( \odot \) The mass fraction of the moon in this case is:
The cosmic abundance of Fe is 0.0016 by mass, so the moon's iron mass fraction is much larger (implying that some process must have preferentially kept iron and discarded other elements like H/He when the moon was forming.

\[ \frac{P_{\text{Fe}}(4R_e^3)}{P_{\text{Fe}}(4R_e^3) + P_{\text{Fe}}(4R_e^3)} = 0.045 \]

3.6. (i) The Fermi pressure is: \( P_F = \frac{51.6}{R^5} \text{ Mbar} \)

The typical internal pressure from hydrostatic equilibrium is:

\[ P_{\text{HSE}} \sim \frac{GM^2}{R^4} \]

Equating these gives:

\[ \frac{51.6}{R^5} \text{ Mbar} \sim \frac{GM^2}{R^4} \]

(ii) Let's get an estimate for \( R_s \) ...
The number of electrons in the object is
\[ n \sim \frac{M}{m_\text{H}} \]  
(since there is 1 electron per nucleon if we assume all hydrogen).

So the average space occupied by an electron is:
\[ \frac{4R^3}{(M/m_\text{H})} \]

Remember that \( r_s \) is dimensionless (defined in terms of \( a_0 \), the Bohr radius), so:
\[ 4(a_0 r_s)^3 \sim \frac{4R^3 m_\text{H}}{M} \]

\[ a_0 r_s \sim \left( \frac{R^3 m_\text{H}}{M} \right)^{\frac{1}{3}} \]

(3) Going back to our equation from (1) gives:
\[ \frac{51.6 \text{ Mbar}}{\sqrt{\frac{R^5}{a_0} \left( \frac{m_\text{H}}{M} \right)^{\frac{5}{3}}} \sim \frac{GM^2}{R^4} } \]
\[ R \approx \frac{51.6 \text{ Mbar}}{G} \frac{M^{1/3}}{m_a^{5/3}} a_0^3 \]

Plugging in values for \( M, m_a, \) and \( a_0 \) gives:

\[ R \approx 10^8 \text{ cm} = \boxed{1000 \text{ km}} \]

(Earth is \( \approx 6,000 \))

4.2

One atom at the center of each edge (12 total), and one at each vertex (8 total) at the center of the cube is \( (0,0) \).

We're considering nearest neighbors only, so only the 12 edge-centered molecules count. These are at a distance from the central molecule of:

\[ |r - (0,0,0)| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}a\right)^2} = \frac{a}{\sqrt{2}} \]
So the interaction energy between any of the edge-centered molecules and the central molecule is:

$$\phi\left(\frac{a}{\sqrt{2}}\right)$$

There are 12 such molecules, but to get the interaction energy per electron, we need to divide by 2 twice:

1. Once to avoid double counting (that's where the factor of $\frac{1}{2}$ in the text eq 1 comes from) → i.e. we don't want to count $\phi_{ij}$ and $\phi_{ji}$ as different energies.

2. Once to get the energy per electron, since each hydrogen molecule in the lattice has two.

So, the interaction potential term is:

$$\frac{12}{2 \times 2 \times 2} \phi\left(\frac{a}{\sqrt{2}}\right) = 3 \phi\left(\frac{a}{\sqrt{2}}\right)$$

Let's figure out the volume per proton. For this part of the problem, I find it
easier to draw the “unit cube” as follows:

(1’ve shifted the cube by \( \frac{a}{2} \) in all three directions so that the grey vertices are now at the centers of the faces).

How many molecules in a cube of side length \( Na \)?

Each “mini-cube” of side length \( a \) has 14 molecules (1 on each of 6 faces, and 1 on each of 8 vertices). However, each face molecule is shared by two unit cubes, so we’ve double counted those. Each vertex molecule is shared by 8 unit cubes, so we’ve octuple-counted those.

So, the volume per molecule is:

\[
\frac{\text{total volume}}{\text{total molecules}} = \frac{Na^3}{\text{total face molts} + \text{total vertex molts}}
\]
\[ \frac{N}{N \left( \frac{6}{2} + \frac{8}{8} \right)} = \frac{a^3}{4} \]

However, what we really want is the volume per proton, so we divide by 2 again and get the factor of 8.

Check out the attached code snippet to see how I estimated the pressure of the phase transition using the Maxwell construction. I got: \( \text{Pran}_0 \approx 0.09 \text{ Ryd/eln} \cdot \text{g/cm}^3 \)

To convert from Ryd/eln to cgs units, we use the following conversion:

\[ 1 \text{ Ryd/eln} \left( \frac{\text{# e/ln}}{1 \text{ g}} \right) \left( \frac{2 \times 10^{-11} \text{ erg}}{1 \text{ Ryd}} \right) \]

The # of electrons per gram of hydrogen is \( \frac{1 \text{ g}}{\text{mp}} = 10^{24} \), so we have:

\[ 1 \text{ Ryd/eln} = 2 \times 10^{13} \text{ erg/g} \]
Using this conversion, our transition pressure (in cgs) is:

\[ 1.8 \times 10^{12} \text{ Ba} = 180 \text{ GPa} \]

\[ \uparrow \]

cgs pressure unit

The actual transition pressure is \(~400 \text{ GPa}\), but we got the right OoM!

6.2

Want: \( \alpha = \frac{1}{T} \left( \frac{\partial V}{\partial T} \right)_P \)

Have: \( C_v, P(e), \gamma \)

Let's use the following Maxwell relationship (eq 1 in the text):

\[ \gamma = \frac{\alpha k}{T} \]

\[ \frac{P e^{C_v}}{C_v} \]

Let's keep all units in cgs throughout the problem! 1 Mbar = \( 10^{12} \text{ Ba} \).
\[ K_T = \left. \frac{\partial P}{\partial e} \right|_T = P \left[ \frac{50}{3} \times 10^{12} e^{2/3} \right] = 16.6 \times 10^7 e^{2/3} \text{ Pa} \]

\[ \gamma = \frac{1}{2} \frac{dK}{dP} - \frac{1}{6} \]

\[ = \frac{1}{2} \frac{dK}{dP} \frac{dP}{P} - \frac{1}{6} \]

\[ = \frac{5}{2} \left[ \frac{50}{3} \times 10^{12} e^{2/3} \right] \left[ \frac{3}{50} e^{-2/3} \right] - \frac{1}{6} \]

\[ = \frac{5}{6} - \frac{1}{6} \]

\[ = \frac{2}{3} \]

Therefore:
\[ d = \frac{\delta e cv}{K_T} \]

\[ = \frac{2 \times c (2) (2 \times 10^8)}{3 \times (50) e^{5/3} \times 10^{12}} \]

\[ = \frac{8 \times 10^{-6} e^{-2/3}} {e^{-2/3} \text{ K}^{-1}} \text{ [P in cgs units]} \]

We have \( cv \) in \( \text{Kb/pmol} \).

To convert to \( \text{erg/g} \),

we multiply by value of \( \text{Kb in cgs} (10^{-16}) \) and divide by \( \text{mp in cgs} (10^{-29}) \)

\[ cv = 2 \times 10^8 \text{ erg/g} \]

In order for \( dT \) to be 0.1,
we must have:

\[ 0.1 = \left[ 8 \times 10^{-6} \rho^{2/3} \right] T \]

\[ \Rightarrow T(\rho) = 12,500 \rho^{2/3} K \quad [\rho \text{ in g s}] \]

Very high temp! This justifies the statement that planetary material don't expand/contract much due to temperature. Along an isentrope, \( \delta = \text{constant by definition}. \) We know that \( K_T = K_S, \) and have a Maxwell relation that

\[ K_S = K_T (1 + \delta \sigma T) \], so \( \sigma T \) must not vary along an isentrope.

The Fermi pressure is \( P = 10 \rho^{5/3} \text{ Mbar}, \)
and the thermal pressure (same as thermal energy density) is:

\[ P_{\text{th}} = P C_v T = \rho \left[ \frac{2K_B}{m_p} \right] T \]
\[ 1.7 \times 10^8 \left( \frac{T}{K} \right) \left( \frac{\rho}{g/cm^3} \right) \text{ bar} \]
\[ = 1.7 \times 10^{-4} \left( \frac{T}{K} \right) \left( \frac{\rho}{g/cm^3} \right) \text{ Mbar} \]

For the thermal pressure to be 10% of the Fermi pressure, we must have:

\[ 0.1 P_F = P_{th} \]

\[ 10 \rho^{5/3} = 1.7 \times 10^{-3} T_F \]

\[ \Rightarrow T = 5,882 \rho^{2/3} \text{ K} \]

This is a pretty high temperature, justifying us ignoring thermal pressure in planets when calculating the EOS.

5.4. a. We have static equilibrium:

\[ \frac{dP}{dr} = -g \rho(r) \]

and are given...
\[ P = P_e \left( \frac{e}{e_e} \right)^{1.4} \]

Differentiating \( \circ \) and plugging into \( \circ \) gives:

\[ \frac{1.4 P_e}{e_e} \left( \frac{e}{e_e} \right)^{0.4} \frac{d \rho}{d \rho} = -\frac{GM_e}{\rho^2} \rho \]

\[ *_{c_i}^2 = \frac{\rho}{e_e} \]

\[ \frac{1.4 c_i^2}{e_e^{0.4}} \int_{e_e}^{\rho} \rho^{-0.6} d \rho = \int_{R_e}^{\rho} \frac{GM_e}{\rho^2} d \rho \]

\[ *_{X_e} = \frac{F}{R_e} \quad *_{x_i} = \frac{R_i}{R_e} \]

\[ \frac{2GM}{7c_i^2 R_e} \]

\[ \left( \frac{\rho}{e_e} \right)^{2/5} = \frac{GM_e}{3.5c_i^2} \left( \frac{1}{\rho} - \frac{1}{R_i} \right) + 1 \]

= \left[ 1 + A \left( \frac{1}{x} - \frac{1}{x_i} \right) \right]^{5/2}

\[ \frac{\rho}{e_e} = \left[ 1 + A \left( \frac{1}{x} - \frac{1}{x_i} \right) \right]^{5/2} \]

b. Mass of envelope is:
\[ M_e = \int_{R_c}^{\infty} 4\pi r^2 \rho(r) \, dr \quad \text{(mass of envelope: negligible)} \]
\[ = \int_{x_t}^{\infty} 4\pi (x R_c)^2 \rho(x) \left[ 1 + \frac{1}{x} \right]^{5/2} R_c \, dx \]
\[ = 4\pi \rho R_c^3 \int_{x_t}^{\infty} x^2 \left( 1 + \frac{1}{x} \right)^{5/2} \, dx \]

We want the value of \( x_t \) for which:
\[ 0.01 = \frac{4\pi \rho R_c^3}{M_e} \int_{x_t}^{\infty} x^2 \left( 1 + \frac{1}{x} \right)^{5/2} \, dx \]

\[ \downarrow \text{Mathematica } \]
\[ \text{(see attached)} \]
\[ x_t = 1.19 \]

\[ \text{\* } \rho_i = 100 \text{ bars and } T_i = 500 \text{ K} \]
\[ \text{so } \rho_i = \frac{P_i (m_{nH})}{k_B T_i} \]
\[ = \frac{(100 \text{ bars}) (2 m_{nH})}{k_B (500 \text{ K})} \]
\[ = 0.0048 \text{ g/cm}^3 \]

\[ \text{\* } c_i^2 = \frac{P_i}{\rho_i} = 2 \times 10^{10} \text{ cm}^2/\text{s}^2 \]

\[ \text{\* } A = \frac{2GM_e}{c_i^2} = 27 \]
Combining hydrostatic equilibrium & ideal gas pressure:

1) HSE: \( \frac{dP}{dr} = -g\rho(r) \)

2) Gas: \( P = \frac{\rho k_B T}{m_{\text{H}_2}} \)

\( m = 2 \) for molecular \( \text{H}_2 \)

\[
\Rightarrow \quad \frac{dP}{dr} = -\frac{GM_c}{r^2} \left( \frac{2m_{\text{H}_2} P}{k_B T} \right)
\]

\[
\int_{10\text{ bar}}^{10\times10^{-3}\text{ bar}} \frac{dP}{P} = -2GM_c m_{\text{H}_2} \frac{r_1}{k_B T} \int \frac{dr}{r^2} \]

\[
\ln(10^4) = \frac{2GM_c m_{\text{H}_2}}{k_B T} \left[ \frac{1}{R_1} - \frac{1}{R_o} \right]
\]

\[
= 3.5A \left[ \frac{R_c}{R_1} - \frac{R_c}{R_o} \right]
\]
\[ 10^y = e^{3.5A \left( \frac{1}{x_1} - \frac{1}{x_2} \right)} \]

mathematica (see attached)

\[ x_2 = 1.35 \]

This corresponds to a radius of:

\[ R_0 = x_2 R_c = 2.2 R_\odot \]

Here's the plot from 4.2:
And the code to generate it:

```python
import numpy as np
import os
import matplotlib.pyplot as plt
from astropy import units as u, constants as cst

r_s = np.logspace(-3 , 2, int(1e3))

r1_specific_vol = ((cst.a0)**3 / cst.m_p).cgs
rydbergs_per_eln2cgs = (u.Ry / cst.m_p).cgs

# metallic hydrogen
E_met = (2.21 / r_s**2 - 2.72 / r_s - 0.2) # [ryd / eln]
V_met = (4 * np.pi / 3) * r_s**3 * r1_specific_vol # [cm^3 / g]

def phi(r):
    return 4000 * (r/np.sqrt(2))**(9)

# molecular hydrogen
E_mol = (3 * phi(r_s) - 1.15) # [ryd / eln]
V_mol = (r_s**3 / 8) * r1_specific_vol # [cm^3 / g]

# plot volume vs energy for both states
plt.figure()
plt.plot(V_met.cgs, E_met, label='metallic')
plt.plot(V_mol.cgs, E_mol, label='molecular')
plt.xlabel('volume [cc/g]')
plt.ylabel('energy [ryd/el]')
plt.xlim(0, 3)
plt.ylim(-1.2, -0.6)

# estimate tangent line slope & use that to get transition pressure (via Maxwell construction)
slope = -0.18 / 2
intercept = -0.925
x = np.linspace(0, 1000, int(1e3))
y = slope * x + intercept
plt.plot(x, y, color='grey', ls='--', label='$\frac{\Delta E}{\Delta V} = {:.2f}$'.format(slope))
plt.legend()
plt.savefig('{}/Desktop/prob42.png'.format(os.path.expanduser('~/')), dpi=250)
```

Here's my mathematica scripts for problems 3.4 and 5.4:
\begin{align*}
V &= pfe = 7 \\
\text{Out[1]} &= 7 \\
V &= pr = 3.3 \\
\text{Out[2]} &= 3.3 \\
V &= R = 1760 \times 10^5 \\
\text{Out[3]} &= 176000000 \\
V &= M = pfe \times 4 \times Rc^3 + pr \times 4 \times (R^5 - Rc^3) \\
\text{Out[4]} &= 28 Rc^3 + 13.2 \{5 \times 451.776 \times 00000000000000000 - Rc^3\} \\
V &= \text{NSolve}[0.391 \times M \times R^2 = 8/5 \times (pfe \times Rc^5 + pr \times (R^5 - Rc^5)), Rc, \text{Reals}] \\
\text{Out[5]} &= \left\{\begin{array}{l}
\text{Rc} \rightarrow -1.75792 \times 10^8, \\
\text{Rc} \rightarrow 4.95706 \times 10^{-7}, \\
\text{Rc} \rightarrow 1.72087 \times 10^8
\end{array}\right\} \\
V &= \text{Rc} = 4.95706 \times 10^{-7} \\
\text{Out[6]} &= 4.95706 \times 10^{-7} \\
V &= (pfe \times 4 \times Rc^3) / (pfe \times 4 \times Rc^3 + pr \times 4 \times R^3) \\
\text{Out[7]} &= 0.045249 \\
\end{align*}