8. Transport Properties

1 Planets have transport

We have dealt so far with equilibrium. Hydrostatic equilibrium is an obvious example\(^1\). Quantum mechanics is also an equilibrium theory in the form we have used it thus far. Thermodynamics is quintessential equilibrium: it provides the state (phase) that a medium prefers to have at a specified P and T\(^2\). However, planets are continuously driven out of equilibrium: Heat flows from one place to another, atoms diffuse, materials flow and sometimes react chemically, electrical currents flow, and some may even have life, a manifestly non-equilibrium state. So it is of great importance to understand the transport properties of planet-forming materials.

It is useful to describe this transport separately for microscopic and macroscopic processes, even though the former can affect the latter. By microscopic, we mean processes that take place at a length scale that is small compared to the distance over which some variable (e.g., temperature) changes significantly. These often involve random walks, with a step size that is a mean free path for atoms or molecules, but with a net outcome (a

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\(^1\) Rotation can be included as a correction to the gravitational potential, provided it only depends on distance from an axis ("rotation on cylinders"). Other flows (e.g., convection) are small compared to the speed of sound.

\(^2\) The correctness of thermodynamics is more subtle than you might suppose. It will often not work at low T or low P and we saw examples of this in Chapter 2. But the insides of planets are high P and T mostly. Even then, you may have an assemblage that is not strictly at a global minimum in the Gibbs sense because of the long diffusion times or because of sustained convection. For example, the upper and lower mantle of Earth need not be in thermodynamic equilibrium with each other (a state that would require them have different compositions according to the Gibbs phase rule) because of some convective homogenization. But if you can specify the local T, P and composition then thermodynamics will usually tell you what phases to expect at that location.
flux of something) arising because of the macroscopic variation (a temperature or compositional gradient). By macroscopic, we mean changes in which the transport arises because of a net flow, as with convection or a river. We have to understand the microscopic before we can deal with the macroscopic, since the large scale flows often depend on the parameters that describe the microscopic response.

Unlike the equilibrium properties, transport properties are often known far less precisely and often lack a similar rigor in their theoretical basis. So errors or uncertainties of a factor of two or worse are expected and fortunately can often be tolerated.

2 Gradients determine fluxes

The central idea in these numerous microscopic processes is that the system is extremely close to equilibrium locally, but that the equilibrium state is very slightly different a short distance away, leading to a flux\(^3\). The following examples, all very famous, are approximations (often called constitutive "laws") and lack the fundamental basis that exists for \( F = ma \) or Newton’s law of gravity:

\[
F = -k\nabla T = -\rho C P \kappa \nabla T \quad \text{(Fourier's law)} \quad (1)
\]

\[
J = -\rho D \nabla C \quad \text{(Fick's law)} \quad (2)
\]

\[
J = -\sigma \nabla v = \sigma E \quad \text{(Ohm's law)} \quad (3)
\]

\(^3\)This way of thinking does not work well for the viscous behavior of solids but works well for all the other example discussed here.
\[\sigma_{ij} = \eta(\partial u_i / \partial x_j + \partial u_j / \partial x_i) \quad \text{(Newtonian viscous flow)} \quad (4)\]

In eq (1), \(\mathbf{F}\) is the heat flux (energy per unit area per unit time), \(k\) is the thermal conductivity (a property of the material) and \(\kappa\) is the thermal diffusivity (units of \(L^2T^{-1}\)). In eq (2) \(\mathbf{J}\) is a mass flux (mass per unit area per unit time) of some constituent present at composition (i.e., mass fraction) \(C\), with diffusivity \(D\) (units of \(L^2T^{-1}\) again). In eq (3), \(\mathbf{J}\) is now current density (amps per unit area; an amp is a Coulomb per unit time), \(\sigma\) is the electrical conductivity and \(V\) is the voltage (i.e., electrical potential). As we shall see in due course, the electrical conductivity is most often seen in planetary problems as a “magnetic diffusivity”, describing the Ohmic decay of electrical currents and associated magnetic field. Eq (4) is noticeably different because the cause of the deviatoric stress \(\sigma_{ij}\) is a gradient of a vector rather than a scalar. Here, \(\eta\) is called the dynamic viscosity and can be written as \(\rho \nu\), whence \(\nu\) is called the kinematic viscosity and has the same units as \(\kappa\) (eq 1) and \(D\) (eq 2) \(^4\).

These relationships are input for some other more fundamental equation. In the first case, it is conservation of thermal energy

\[\rho C_P \partial T / \partial t = -\nabla \cdot \mathbf{F} \quad \rightarrow \partial T / \partial t = \kappa \nabla^2 T \quad (5)\]

In the second case, conservation of the mass of the constituent \(C\) yields an identical diffusion equation involving \(D\) instead of \(\kappa\) \(^5\). There is a similar diffusion equation for momentum but we will not need it here, and a similar diffusion equation for magnetic field that we will meet much later.

\(^4\)Eq 4 lacks the negative sign of all the others but that is just matter of convention and not significant

\(^5\)The results quoted here that lead to the Laplacian operator only apply when the material constants, e.g. \(\kappa\), are independent of position. This assumption is often violated.
3 Random Walks and a simple View of Diffusion

Equation (5) and its compositional counterpart suggest by dimensional analysis that there is a characteristic length scale associated with diffusion, either \((\kappa t)^{1/2}\) or \((Dt)^{1/2}\). This called the diffusion length for heat and composition respectively. Trivially, we can also invert these and talk about a diffusion time \((L^2/\kappa\) and \(L^2/D\) respectively), where \(L\) is some length scale of interest such as the thickness of a planetary layer or even the planet radius. The diffusion length can be easily understood by considering the outcome of placing a spike (delta function) of temperature or composition at the origin at time \(t\) and asking how this spreads over time. A Gaussian results in the simplest case, with a characteristic width that is about the diffusion length.

An example of a 2D random walk is shown below (after 25,000 steps).

It is useful to review a very simple result of random walks (detailed in the first Appendix). Suppose we take \(N\) steps, each of length \(a\), but in random directions. We do this many times. The resulting displacement from the
origin for a given case is then $\mathbf{r}$, where

$$\mathbf{r} = \sum_{i=1}^{N} \mathbf{r}_i; \quad |\mathbf{r}_i| = a \quad (6)$$

but this means

$$\mathbf{r}.\mathbf{r} = Na^2 + \text{a term that must average to zero over many sample sums} \quad (7)$$

In short, the mean displacement from the origin after $N$ steps is $N^{1/2}a$. This is the characteristic of all random walks$^6$.

Now suppose a single step is done at speed $v$ so that it takes time $a/v$. Then after a total time $t$ of $N$ timesteps ($t=Na/v$), the mean distance from the starting point is $N^{1/2}a = (vt/a)^{1/2}a = (Dt)^{1/2}$ where $D=av$. Motivated by the kinetic theory of gases we can think of diffusion as involving a step size of order $10^{-8} cm$ (the interatomic spacing inside a planet) and a velocity of about $10^6 cm/s$ (the sound speed). This suggests a diffusivity of $10^{-2} cm^2/s$. This is actually about right for the diffusion of heat in an insulator$^7$. In fact, the naive way to think about this kind of diffusion is to imagine that all the diffusive quantities are then comparable:

$$\kappa \approx \nu \approx D \approx 10^{-2} cm^2/s \quad (8)$$

Of course, this is simplistic. Thermal diffusion in a metal is accomplished by the free electrons, which move much faster than atoms. Even so, you gain only about one order of magnitude. The diffusion of atoms in a liquid is typically somewhat less, perhaps more frequently $10^{-4} cm^2/s$. It is far less for a solid (discussed later). But even for the larger value, we find that the diffusion length is small.

$^6$See the Appendix for more insight into the range of outcomes.

$^7$Somewhat fortuitously. Heat diffusion is then the scattering of phonons; there is no need to physically transport an atom to a new location since you are merely passing on energy
4  The Inefficacy of Diffusion

Consider an elapsed time of the age of the solar system. In seconds, that is $(4.5 \times 10^9)(3 \times 10^7) \approx 10^{17} \text{sec}$. For a diffusivity of $10^{-2} \text{cm}^2/\text{s}$ this implies a diffusion length of order $(10^{15} \text{cm}^2)^{1/2}$ which is about 300km! This is small compared to the size of most planets.

So diffusion of heat in insulators is often unimportant on the planetary scale. It is useful to consider this: Suppose I let off a small explosion at the center of Earth. The seismic (sound wave) signal from this will arrive at Earth’s surface in about 20 minutes. The thermal effect of this event (assuming only diffusion and no convection) would not be discernible at Earth’s surface after even the age of the universe! It would scarcely even get out of the core. This inability of diffusion to get rid of heat lies at the heart of why planets convect and why we got a small size for planets in Chapter 1.

An example of where diffusion would matter is in small bodies (e.g., asteroids or small icy satellites or Kuiper belt objects). Even in small bodies, you need to ask whether the relevant time scale is the age of the solar system or some other time scale, e.g., the decay of a short-lived isotope such as $^{26}\text{Al}$ (half life of order one million years). Thus, a body larger than about 5 or 10km, the diffusion length for 1 million years, might be expected to get very hot internally if it formed very early in the solar system. And as we shall see later, diffusion still matters in boundary layers and for that reason figure prominently in our understanding of convection, especially for high viscosity bodies (e.g., terrestrial planetary mantles).

Diffusion of atoms is less important, because the relevant diffusivity is typically even smaller. It is almost universally true that large scale transport of atoms in a planet is accomplished by fluid dynamical processes (convection) and not by diffusion.

Diffusion of heat in a metallic core is obviously somewhat more important because of the higher thermal conductivity of a metal. It is even important in Earth’s core (despite its size). But diffusion is not important in Jupiter’s core (despite being a metal).

5  What about Activation?

In the case of molecular diffusion we have imagined a process of uninhibited translational motion, analogous to the kinetic theory of gases. In reality,
atoms or molecules of a foreign species embedded in a solid must jump out of some energy minimum in order to migrate. In a liquid, their motion may also be impeded if the structure is polymeric (long chains of atoms or molecules loosely linked together, as if it were a giant molecule). This lowers the diffusivity. The easiest case to think about is thermally activated jumping from one site to another. In accordance with Boltzmann statistics, the time it takes to make a jump can be longer by $\exp(-\Delta H/k_B T)$ where $\Delta H$ is then enthalpy change required for the jump. This changes the diffusivity by a similar (Arrhenius) factor:

$$D = D_0 e^{-\Delta H/k_B T}; \quad \nu = \nu_0 e^{\Delta H/k_B T} \quad (9)$$

where the prefactors are similar to our previous unimpeded estimates.

6 What about Radiation?

Heat can also be carried by photons. Since they can scatter and be absorbed then re-emitted, they can be represented by a temperature-dependent “conductivity” (i.e. Fourier’s law) provided the mean free path is small compared to the distance over which $T$ (for example) changes significantly. This proviso is violated in planetary atmospheres (one of the reasons why radiative transfer is such a big and complex subject) but not in planetary interiors (because of their much higher density). Even so, the mean free path of photons can often be much larger than the mean free path of atoms or phonons, especially near visible wavelengths, since atoms do not absorb efficiently then. (That’s why you can see through Earth’s atmosphere but it’s relatively opaque at both IR and UV). And the $T$ inside planets can be many thousands, corresponding to the visible range (which is defined by the Sun’s temperature of about 6000K). So it is natural to speculate that the deep mantle of Earth (for example) might be capable of carrying significant heat by radiation.

One consideration limiting radiative transport is the energy in the radiation field, which is much smaller than the energy in the atomic vibrations.

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$^8$Enthalpy $\Delta H = \Delta E + P\Delta V$ includes the work that might be needed to change the density.

$^9$This complementarity is common but not universal. In the case of solids, the mechanism for "viscosity" can be unrelated to the bulk diffusion process.
But purity matters more. Pure crystals can indeed be quite transparent (sometimes) but real planetary material is usually dirty. Even small amounts of dirt (as rare phases or inclusions or interstitials or whatever) can matter. In this respect, radiative opacity (inversely related to conductivity) is unlike thermodynamic parameters or transport parameters that depend on phonons. The evidence at present suggests that radiative transport is not important\(^\text{10}\).

How can we get high thermal diffusivity? In metals it would come from free electrons. In insulators, a key consideration is anharmonicity (the failure of the vibrations to be simple undamped harmonic oscillators). This means that a material with high Debye temperature is a good choice. Diamond is an example. But planets are hot, even approaching or exceeding the melting temperature inside. The consequence is short mean free paths and the typical diffusivities we estimated above.

## 7 A Simple Model for Electrical Conductivity

In a conductor, free charges are accelerated by the presence of an electric field but after a relaxation time \(\tau\) they lose that momentum because of scattering by a phonon. This is called electron-phonon scattering and is the prime mechanism for limiting both electrical and thermal conductivity in metals\(^\text{11}\). The velocity attained by the charge \(e\) is accordingly \(eE\tau/m\) where \(m\) is the mass of the particle, and this drift velocity\(^\text{12}\) yields a current density \(J = nev\), whence \(J = \sigma E\), which is Ohm’s law with a conductivity given by

\[
\sigma = ne^2\tau/m \quad (10)
\]

\(^{10}\)Radiation certainly does not matter if there are any free electrons. For example, metals are extremely opaque. Even a modest presence of free electrons can change the opacity enormously.

\(^{11}\)The same basic idea applies to liquids though there are no phonons, just disordered structure that disrupts the trajectory freely moving charges.

\(^{12}\)The charges are moving in all directions but there is systematic ”drift” arising from the imposed electric field. Memory of the motion caused by the field is lost at each scattering event. The parameter \(\tau\) is just a convenient way of quantifying the scattering. If the charges are electrons then its evaluation requires quantum mechanics even though the theory looks classical.
Notice that the sign of $e$ does not matter. In a metal, $m$ is the electron mass and $n$ counts only the free (unbound) electrons. In SI units, typical numbers for a good metal (e.g., Cu at room temperature) would be $n \approx 10^{29} m^{-3}$ and $\tau \approx 10^{-14} \text{sec}$. Together with $e = 1.6 \times 10^{-19} \text{Coulombs}$ and $m = 9.1 \times 10^{-31} \text{kg}$, this gives $\sigma \approx 10^7 \Omega^{-1} \text{m}^{-1}$. (This is usually written $S/m$ where $S$ stands for Siemens, which is the same thing as inverse Ohm. Don’t ask me why it was considered necessary to give the inverse of something a special name!)

In a semiconductor you can still use eq(10) but $n$ is smaller by a factor of $\exp[-\Delta E/2k_BT]$ where $\Delta E$ is the band gap energy and the factor of two in the denominator arises from fundamental statistical mechanics, according to which the product of electron density and hole density scales as $\exp[-\Delta E/k_BT]$ and these densities must be equal. This is presumably relevant to Jupiter (cf. chapter 4), as well as to electromagnetic sounding in terrestrial planets and the coupling of winds to magnetic field in hot Jupiters.

In a metal, the “relaxation time” $\tau$ is related to the Fermi velocity, but depends on electron-phonon scattering.... the same process that determines thermal diffusion provided electrons are most important for heat transport (usually true for a metal). So it is not surprising that there is a fundamental relationship between thermal and electrical transport in metals called the Wiedemann-Franz law

$$k/\sigma T = 2.45 \times 10^{-8} \text{ (SI units)} \quad (11)$$

This turns out to be important for understanding magnetic field generation since a very high electrical conductivity implies a very high thermal conductivity, which can turn off field generation by a dynamo because convection may not happen.

8 Solid Rheology

For economy of words, ”stress” in this section means deviations from hydrostatic pressure. Idealized solids are elastic, which means that the stress-strain relationship is reversible (i.e., time-independent) and obeys Hooke’s law. Real solids have imperfections, often due to obvious things such as
granularity or impurities but sometimes due to thermally or stress-activated things such as dislocations. Rheology is the general term for how a solid responds to stress. It is complicated, but part of the response may be represented by viscous-like behavior at long time scales. At shorter times (e.g., tides) you need to consider a more complicated response, for example, viscoelasticity (described further below) and at large enough stress you may need to worry about failure or plasticity (effectively a low apparent viscosity but very stress-dependent and often history dependent). It is sometimes insufficiently emphasized that the existence of an algebraic relationship between stress and strain (or strain rate) is an assumption and not always true, so the rheology of real solids can be very complicated. Moreover, it is extremely difficult to attain the conditions of planets in the lab, with the consequence that the quantification is difficult and theory is often extrapolated well beyond the range of measurements.

Terrestrial planets and large icy bodies convect in their solid regions. This arises because of the inability to get rid of heat by diffusion, as described in this chapter. In effect, what happens usually is that because of the inefficiency of diffusion, the planet stays hot and adopts an internal temperature for which the needed heat release is accommodated. This is conceptually very different from the way one might think about heat conduction or radiation: The planet chooses its viscosity! This can work often because the viscosity is usually very strongly temperature-dependent so one only needs to adjust the temperature a small amount. But it is often unsatisfying since we would like to know whether the planet can actually choose a value that works and avoids melting, for example. As a consequence, it is necessary to look at and think about how the rheology really works, even though one should never think of it as a set of parameters that could be "looked up", the way you might look up the specific heat or thermal conductivity. In later chapters we will see multiple examples of where some understanding of rheology is needed to interpret the way planets behave.

Under stress, solid material may undergo irreversible strain (meaning that the strain is finite even when the stress is reduced back to zero). At fixed stress, there may (in steady state) be some finite strain rate. This is observationally true: In polycrystalline materials in the lab, one typically finds something like this:
\[
\frac{de}{dt} = A \sigma^n d^{-m} e^{\left[ -\Delta H/k_B T \right]} \tag{12}
\]

where \( e \) is the strain\(^{13} \), \( \sigma \) is the (non-hydrostatic) stress, \( d \) is the grain size, \( n \) and \( m \) are often (but not necessarily) integers, and \( A \) is some constant (usually insensitive to temperature and pressure but possible dependent on other things such as oxygen fugacity). Remember that strain rate \( \frac{de}{dt} \) may be thought of as a velocity gradient (cf, Eq 4). If one tries to define a viscosity (stress divided by strain rate), one gets

\[
\eta = A^{-1} d^m \sigma^{-(n-1)} e^{\left[ \Delta H/k_B T \right]} \tag{13}
\]

but this is only independent of stress in the special case \( n=1 \) and \( d \) is independent of stress (the Newtonian fluid case). We can always define an effective viscosity for some flow but it’s important to remember that in general the flow is non-linear (i.e., stress and therefore effective viscosity varies from place to place), so that this is a definition of convenience rather than a correct characterization of the rheology.

In solids, the two most important mechanisms for creep are likely to be Nabarro-Herring (diffusion) creep, for which \( n=1 \), \( m=2 \) and dislocation creep, for which \( n \approx 3 \) or more, (but may not be an integer or even constant) and \( m=0 \). To the extent that the grain size also depends on stress, it could be argued that there is never any truly Newtonian creep. Generally, non-Newtonian creep will dominate at high stress, because \( n > 1 \) for that process and the mechanism that predicts the largest strain rate will tend to dominate. In either case, one expects that the activation energy is much larger than \( k_B T \). In fact, it is often found that the parameter inside the exponential is rather similar among different materials provided it is evaluated at the melting point. This makes sense if one thinks (naively) of the melting point as being high in materials where it is difficult to make voids or diffuse atoms and low when it is easy. The following table quantifies this:

\(^{13}\)For convenience, the tensoral character of stress and strain are omitted, but of course they must be considered in a real calculation.
(Note: There is of course no difference between using $RT_m$ and $k_B T$ in evaluating the RHS column, provided one remembers that the enthalpy is per mole in the first case and per atom or molecule in the second case).

The following figures show (crudely) two mechanisms for creep. The first figure shows how atoms can diffuse away from a region of high stress (compression) to a region of low stress (extension) thereby allowing the block of material to continuously deform. This is possible because all materials have a finite vacancy level at finite temperature. It is a linear (i.e., Newtonian) mechanism because the process is thermally activated at all stress levels.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\Delta H/RT_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silicates &amp; oxides</td>
<td>25 to 30</td>
</tr>
<tr>
<td>water ice</td>
<td>~25</td>
</tr>
<tr>
<td>metals</td>
<td>15-20</td>
</tr>
</tbody>
</table>

This diffusion creep depends on grain size since the diffusion is to the grain boundaries.

The next figure illustrates several layers of atoms. In the process of dislocation climb, a truncated layer of atoms connects to an adjacent layer, thereby shifting the dislocation (and the material) over. By a sequence of such steps there will be a net strain. Although dislocations are thermally
activated, the stress required is finite and the overall process is nonlinear. (A
and B indicate a line of atoms perpendicular to the plane of the figure).

8.1 Homology

The table above suggests that we might be able to write an effective viscosity
in the form

$$\eta = \eta_0 \exp[g(T_m/T - 1)]$$

so that $\eta_0$ is the viscosity at the melting point (possibly pressure-dependent)
$T_m$ and $g$ is dimensionless. Our table gives $g$ (approximately). The ratio of
$T/T_m$ is known as homologous temperature. In a polycrystalline assemblage,
one must usually interpret $T_m$ as the melting point of the major phase... the
one that dominates in the deformation (e.g. olivine in Earth’s upper mantle).
So $T_m$ is not the solidus! This makes sense, since the ability to form a low
melting point component is usually governed by minor constituents that are
not inside the olivine crystals. (Actually, this is an oversimplification; for
example, the presence of water lowers the viscosity of olivine and also lowers
the solidus.) Of course, $\eta_0$ is not independent of the stress level associated
with the convection.

Homology is neither right nor completely wrong. It is "right" in the sense
that it captures the expected correlation between activation energy and the
energy scale relevant to melting. It is "wrong" in the sense that it tends to
simplify a situation that is actually very complicated and could even mislead
you by an order of magnitude or even more.
Roughly speaking, convective stress levels are often of order $\rho g \alpha \Delta T \delta$ where $\Delta T$ is the temperature fluctuation driving the convection and $\delta$ is some characteristic length scale associated with these motions (e.g. boundary layer thickness, not the depth of convection). We will discuss this more in a later chapter. For Earth, this gives (3).(1000).$(2 \times 10^{-5}).(500K).(10^7) = 3 \times 10^8 cgs = 300 \text{ bars}$ (but actually 100 bars is a better mean value). For Ganymede, this gives (1).$(160). (1 \times 10^{-4}). (20K). (2 \times 10^6) = 0.2 \text{ bars}$.

With this parameterization, one finds roughly the following things:

(i) In terrestrial mantles, $g \approx 30$; $T_m \approx 1950K$ and $\eta_0 \approx 10^{18}$ Poise at a stress level of around 100 bars. This means that $\eta \approx 10^{21}$ Poise $(10^{20}$ Pa.s) at the uppermost mantle condition of $T \approx 1600K$. Actually, observations suggest a value that is about one order of magnitude lower for Earth. This is believed to be due to “water” (dissolved hydrogen in the olivine crystals).

(ii) In water ice, $g \approx 25$; $T_m = 273K$ and $\eta_0 \approx 10^{15}$ Poise at $\sigma \approx 1 \text{ bar}$.

(iii) It is important to remember that because viscosities can vary over many orders of magnitude, it is easy to get (or assume) the “wrong” value by an order of magnitude or more.

### 8.2 Additivity

In general, we expect the strain rates of different mechanisms to be additive, i.e.,

$$\frac{de}{dt} = \left(\frac{de}{dt}\right)_{\text{diffusion}} + \left(\frac{de}{dt}\right)_{\text{dislocation}} \quad (15)$$

with the result that the dominant mechanism will change as stress changes. In terrestrial mantles, the stress level at which diffusion (potential Newtonian creep) begins to dominate depends on grain size and mineral phase. Most probably it is around 10 bars or even less. In the lower mantle, Newtonian creep may be more important but this is not well known. In ice, the transition is probably around a bar or a little less. It follows that most planetary bodies are non-Newtonian in most places. Remember also that since grain size depends on stress, even a Newtonian mechanism can yield a non-Newtonian response.
8.3 Viscoelasticity

Solid planets have outer regions that are so cold (e.g., temperature less than half the melting point) that their response is not viscous even on geologic timescales. These regions may respond elastically for modest stresses but then fail at stresses exceeding a few hundred bars to a few kilobars (rock) or a few tens of bars or less (ice). In some cases, it is necessary to consider a more complete rheological description, one where the material has both viscous and elastic properties depending on the timescale on which it is stressed. We will return to this when we talk about lithospheres and when we consider how satellites respond to tides (Chapter 17).

8.4 Effect of Pressure

If you have homologous scaling, lines of constant homologous temperature $T/T_m$ (i.e., parallel to the melting curve) would be isoviscous. Since the melting curve is generally steeper than the adiabat, viscosity should increase with depth in planets (after you get into the adiabatic region). This is widely thought to be true, though subject to the complications of phase transitions. In reality, the argument based on homologous temperature is only a qualitative guide and should not be used quantitatively... if you look back at the more fundamental equation, you see that what really matters is the activation volume in the $\Delta H$ that governs the rheology. Nonetheless, activation volumes are often qualitatively like the volumes to create melting, e.g., the activation volume for Ice I is negative and the melting point for Ice I decreases with pressure. Most activation volumes are positive in materials with positive melting point gradients.

8.5 Effect of Partial Melting

When a polymineralic material undergoes partial melting, several factors influence the viscosity of the medium as a whole. For example, the melt may lower the solid-solid contact area of the grains, thus increasing the deviatoric stress (at the scale of individual grains) and increasing the strain rate. In other words, the viscosity will go down. In some cases, this effect is large; an order of magnitude even at modest melt fraction. In general, the effect is complicated, however. An example of an opposite effect: The melt may be the favored location for molecules that weaken the crystal (e.g., water in
olivine). In this circumstance it is even conceivable for the viscosity to go up. As an additional complication, the melt may migrate (under the action of gravity, given it has a different density from the solid). We will discuss this in a later chapter. It has sometimes been argued that Earth’s asthenosphere (the weak upper most part of the mantle) owes its existence to partial melt, but it might in part be the weakening of olivine arising from water molecules within the crystals.

*Summary* Planets have transport. To understand this, you must consider microscopic processes such as conduction, if only to demonstrate that they are often ineffective, implying the need to consider fluid dynamics. The microscopic processes obey constitutive relationships in which the flux depends on the gradient of some macroscopic quantity (e.g., temperature or composition). The underlying physical mechanism usually depends on a random walk. In planets, the step size of that random walk is so small compared to the planet radius that the resulting diffusion can be incapable of delivery even on a timescale of the age of the universe. Heat created at the center of Earth would take $10^{12}$ years to escape if diffusion were the only mechanism. Hot solids can flow and exhibit ”viscosity”. Planets can often ”choose” their viscosity by adjusting their temperature but the underlying physical mechanisms are imperfectly understood and the ability for a planet to melt or avoid melting depends on parameters whose values are sensitive to temperature and affected by stress and grain size.
Appendix

Consider a 1D random walk consisting of two equal choices at each timestep: A step (+1) to the right or a step (-1) to the left. After n steps the expected (mean) position is zero but there is variance about that position of about \( n^{1/2} \). We shall now prove this and derive the Gaussian representing that variance. From elementary probability theory, the probability that you will have n-k positive steps and k negative steps is given by the (un-normalized) parameter P:

\[ P = \frac{n!}{(n-k)!k!} \]  

(16)

Since n and k are assumed to be huge, we can use the Stirling approximation \( \ell n(n!) \approx n \ell n(n) - n \). It follows that

\[ \ell n(P) = -n[x \ell n(x) + (1 - x) \ell n(1 - x)] \]  

(17)

where \( x = k/n \). (Not coincidentally the expression is mathematically the same as the mixing entropy of two species with \( x \)-mole fraction.) The maximum of this function is at \( x = 1/2 \) as expected (the most favorable outcome). If we write \( x = 1/2 + \delta \) then we find that the \( \delta \)-dependent part of P is given by

\[ \ell n(P) \propto -2n\delta^2 \]  

(18)

which is a Gaussian for P about \( k=n/2 \) with a variance of about \( k^{1/2} \) in k (or \( n-k \)). This can be trivially generalized to multiple independent dimensions.
Problems

8.1) In hydrogen planets more massive than Jupiter, the radius $R$ decreases with increasing mass $M$, $R \propto M^{-1/3}$. At the same time, the thermal diffusivity $\kappa$ (and other diffusivities) might be expected to change in accordance to the scaling discussed in this chapter, i.e., particle spacing times sound speed. What does this suggest about the dependence of characteristic diffusion time $R^2/\kappa$ as a function of mass (all the way up to the masses of white dwarfs 1000 Jupiter masses).

**Solution:** $\kappa \propto v.a \propto (K/\rho)^{1/2} \rho^{-1/3} \propto \rho^0$ because $K \propto \rho^{5/3}$ in that limit. So the only effect on $R^2/\kappa$ is in $R$. This can be a large effect when one gets up to white dwarf masses (objects the radius of Earth). Even then, it would seem that diffusion times will be longer than the age of the universe unless $\kappa$ is substantially larger than this scaling suggests (i.e., $>> 0.1 cm^2/sec$). The shortcoming of this argument is that it does not explicitly address the scaling for heat conduction by electrons in a metal. This can be large if the body is cold enough; and this depends in turn on the ratio of actual to Debye temperature. To appreciate this effect, we would have to analyze the thermal structure of the body.

8.2) Shown here are the Livermore shock wave experimental results for the electrical conductivity of $H_2O$ and $H_2$, as a function of pressure (but temperature also changes of course). Under the highest pressure conditions of these experiments, the temperature is several thousand degrees. The units of conductivity are (ohm.cm)$^{-1}$.
(a) Show that the high pressure results for water are consistent with the formula derived in this chapter, provided it is assumed that the charge carriers are protons, there is about one free proton per water molecule, and the free protons go about one intermolecular spacing before colliding. You can assume the protons have a thermal velocity, with $T=5000$K. The water density is about 2 g/cc. You’re only looking for approximate consistency, not exact reproduction of the observed value.

(b) Try to do the same thing for the hydrogen results, but here the question becomes: What fraction of the electrons are “free”? To get the relaxation time, you should assume that the electrons move at a Fermi velocity $v_F$, and scatter after one intermolecular spacing. (Fermi energy can be written as $mv_F^2/2$, where $m$ is the electron mass.) Hydrogen density is about 1g/cc. (The fact that some experimental work is on $D_2$ is irrelevant because electrons are the current carriers, but when I say hydrogen density, I mean $H_2$ of course).

Important: You can easily get into trouble with units in this problem (as is so often the case in electromagnetism) For historical reasons, conductivities are often given in non-standard units, and $(\text{ohm.cm})^{-1}$ is not an SI unit, nor is it esu or emu units! To calculate consistently in SI units, use $e$ (charge of electron or proton) = $1.6 \times 10^{-19}$ Coulombs, $n$ is per cubic meter, $m$ is in kilograms ($1.6 \times 10^{-27}$ kg for a proton) and the answer will be in $(\text{ohm.meter})^{-1}$ or equivalently S/m.

8.3) Heat transport by radiation is $F = d(\sigma T^4)/d\tau$ where $\sigma$ is the Stefan-Boltzmann constant and $\tau$ is optical depth. If we write $d\tau = \rho \kappa dz$ where $\rho$ is density, $\kappa$ is the opacity (not to be confused with thermal diffusivity, which has different units) and $z$ is depth, then one mean free path $\ell$ of a photon is by definition $1/\rho \kappa$. So we then get $F = 4\ell \sigma T^3 dT/dz$ for radiative transport. What is the predicted radiative transport deep in a planet as a function of the mean free path of the photons that carry the heat? Assume $T=4000$ and $F =0.3$K/km (reasonable for deep Earth). Your answer will be in terms of $\ell$.

**Solution** We have $F = (4)(6\pi)(5.7\times10^{-5})(4000)^3(3\times10^{-6}) \approx 400\ell$ cgs which is comparable to the actual heat flow (80 cgs) if $\ell$ is a macroscopic

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$^1$The formulation here is approximate. In reality one must average the opacity over the Planck function for the photon frequencies you expect for a specific temperature. A transparent window in frequency space can change everything, even if the window is narrow.
8.4) The goal here is to understand whether the center of a small body will undergo melting because of a short-lived heat source. The obvious example is $^{26}$Al in meteorite parent bodies, some of which partially melted to form iron cores and thus allow for iron meteorites. We consider a body of radius $R$ and melting point $T_m$.

(a) We first must show that accretional heating does not matter for small $R$. Find the maximum possible accretional temperature rise for a 50km sized silicate body (density 3.3 g/cc and specific heat of $8 \times 10^6$ erg/g.K) by equating the gravitational energy of formation to the stored heat.

(b) Having found an answer that is small compared to $T_m$ we now ignore that effect. We further assume the surface temperature is maintained at $T=0$. (We can always redefine the temperature scale to make that simplification; $T_m$ below can be modified from its true value if necessary to accommodate that.) We assume the heating source is exponentially decaying with time, appropriate to radiogenic heating. In other words, the thermal diffusion equation for this (assumed solid) body is

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + A T_m \exp\left[-\frac{t}{\tau}\right]/\tau$$  

(19)

with $T(r, t=0)=0$ and $T(r=0,t)=0$; where $\kappa$ is thermal diffusivity, $A$ is a dimensionless number and $\tau$ is a decay time. Show that if $A < 1$ then it is never possible to get melting at the center no matter what the value of $R$ is. (If this takes you more than a line or two then you are doing it wrong. Obviously this will not apply if $R$ is large enough because we ignored accretion.)

(c) By ignoring $\frac{\partial T}{\partial t}$ on the LHS (this is called the “steady state” solution even though it is time-dependent) show (by solving the equation) that

$$T(r,t) = A T_m (R^2 - r^2) \exp(-t/\tau)/6\kappa \tau$$  

(20)
(NB. It is physically unrealistic because you will notice that it does not satisfy \( T(r,0) = 0 \); that’s the price you pay for “steady state”. But it would be correct nearly everywhere once enough time has elapsed and the body is small enough.) Clearly you must have \( A > 6\kappa\tau/R^2 \) for melting, though the equation does not tell you by how much. If your first attempt does not give you the coefficient of 6 it is probably because you’ve forgotten how to do spherical geometry for \( \nabla^2 T \).

(d) \( R = (6\kappa\tau)^{1/2} \) is clearly an important (“critical”) length scale in this problem and it should be obvious to you that bodies small compared to this can never melt unless \( A \) is much larger than unity. For plausible \( A \) (of say about 2) it a reasonable first guess of how big you have to be. Calculate that critical radius for the plausible values of \( \kappa = 0.01 \text{ cm}^2/\text{s} \) and \( \tau = 1 \text{ million years} \). [The full solution of this problem is not simple. See Carslaw and Jaeger, p245. But you should get 10 or so km and that is about right. More typically a few tens will do it. Even less for a comet that forms very early, since \( A \) is larger because \( T_m \) is much smaller (assuming you are only seeking to separate the rock from the ice). The key to this “small” answer is of course the small timescale, plus a lot of heat.]